MISCELLANEOUS EXAMPLES

Very Short Answer : [1 Mark]

1. Let * be a binary operation defined by a* b = 2a + b – 3. Find 3*4. **[CBSE 2008, 1M]**

Sol. $a * b = 2a + b - 3$

$\therefore \quad 3 * 4 = 2(3) + 4 - 3 = 7$ **[1]**

2. Let * be a binary operation on N given by a * b = HCF (a, b), a, b ∈ N. Write the value of 22 * 4. **[CBSE 2009, 1M]**

Sol. 22 * 4 = H.C. F of (22, 4) = 2 **[1]**

3. If $f : R \to R$ be defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$. **[CBSE 2010, 1M]**

Sol. $f : R \to R$

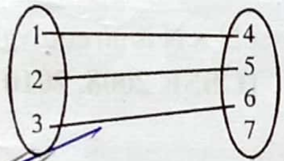
$f(x) = (3 - x^3)^{1/3}$

$fof(x) = f(f(x)) = \left[3 - \{f(x)\}^3\right]^{1/3}$

$= \left[3 - \left\{(3 - x^3)^{1/3}\right\}^3\right]^{1/3} = \left[3 - (3 - x^3)\right]^{1/3} = x$ **[1]**

4. Let A = {1,2,3}, B = {4,5,6,7} and let f= {(1,4), (2,5), (3,6)} be a function from A to B. State whether f is one one or not. **[CBSE 2011, 1M]**

Sol. Yes f is one-one



[1]

5. The binary operation * : R × R → R is defined as a * b = 2a + b. Find (2 * 3) * 4. **[CBSE 2012, 1M]**

Sol. Given a*b = 2a + b

(2*3) * 4 = (4 + 3) * 4 = 7 * 4 = 14 + 4 = 18. **[1]**

6. If R = {(x, y) : x + 2y = 8} is a relation on N, write the range of R. **[CBSE 2014, 1M]**

Sol. Given $x + 2y = 8 \quad x, y \in N$

$y = \frac{8-x}{2}$

at $x = 2 \quad y = 3$

$x = 4 \quad y = 2$

$x = 6 \quad y = 1$

R = {(2, 3) (4, 2) (6, 1)}

Range of R = {3, 2, 1} **[1]**

Short Answer : [4 Marks]

7. Determine which of the following binary operations on the set N are associative and which are commutative.

(a) $a*b = 1 \ \forall \ a, b \in N$ (b) $a*b = \frac{a+b}{2} \ \forall \ a, b \in N$ **[CBSE 2006, 4M]**

Sol. (a) a * b = 1 and b * a = 1

a * b = b * a = 1 ∀ a, b, ∈ N

∴ Binary operation is commutative **[1]**

Again, $(a * b) * c = 1 * c = 1$

and $a * (b * c) = a * 1 = 1 \ \forall\ a, b, c \in N$

$\therefore (a * b) * c = a * (b * c)$

$\therefore$ Binary operation is associative **[1]**

(b) $a * b = \dfrac{a+b}{2} = \dfrac{b+a}{2} = b * a$

$\therefore$ Binary operation * is commutative **[1]**

Now, $(a * b) * c = \left(\dfrac{a+b}{2}\right) * c$

$$= \frac{\frac{a+b}{2}+c}{2} = \frac{a+b+2c}{4}$$

and $a*(b*c)$

$$= a * \frac{b+c}{2} = \frac{a+\frac{b+c}{2}}{2} = \frac{2a+b+c}{4}$$

$\therefore (a * b) * c \neq a * (b * c)$

Hence binary operation* is not associative. **[1]**

8. Show that the relation R defined by $(a, b) R (c, d) \Rightarrow a + d = b + c$ on the set $N \times N$ is an equivalence relation. **[CBSE 2008, 2010 4M]**

Sol. (i) Reflexive :

Let $(a, b) \in N \times N$ such that

if $(a, b) R (a, b) \Rightarrow a + b = b + a$

which is true $\forall\ (a, b) \in N \times N$

$\therefore$ R is reflexive **[½]**

(ii) Symmetric :

Let $(a, b), (c, d) \in N \times N$ such that

if $(a, b) R (c, d) \Rightarrow a + d = b + c$

$\Rightarrow b + c = a + d \Rightarrow c + b = d + a$

$\Rightarrow (c, d) R (a, b)$

$\therefore$ R is symmetric **[1½]**

(iii) Transitive :

Let $(a, b), (c, d), (e, f) \in N \times N$ such that

if $(a, b) R (c, d) \Rightarrow a + d = b + c$ ....(i)

& $(c, d) R (e, f) \Rightarrow c + f = d + e$ ....(ii)

from equation (i) + (ii)

$\Rightarrow a + d + c + f = b + c + d + e$

$\Rightarrow a + f = b + e \Rightarrow (a, b) R (e, f)$

$\therefore$ R is transitive. **[1½]**

Since R is reflexive, symmetric and transitive on $N \times N$.

Hence R is equivalence relation. **[½]**

9. Let : $N \to N$ be defined by $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if n is odd} \\ \frac{n}{2}, & \text{if n is even} \end{cases}$ for all $n \in N$ **[CBSE 2009, 4M]**

Find whether the function f is bijective.

Sol. $f(n) = \begin{cases} \frac{n+1}{2} & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$

for f to be bijective it should be one-one and onto [1]

Let $x_1 = 3$ & $x_2 = 4$

here $f(3) = \frac{3+1}{2} = 2$ & $f(4) = \frac{4}{2} = 2$ [1]

$\because$ $f(3) = f(4)$ but $3 \neq 4$

$f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$ [1]

i.e. f is not one-one

Hence f is not bijective. [1]

10. Show that the relation S in the set $A = \{x \in Z : 0 \le x \le 12\}$ given by $S = \{(a,b) : a,b \in Z\}, |a-b|$ is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. **[CBSE 2010, 4M]**

Sol. (i) Reflexive :

Let $a \in A$

$|a-a| = 0$ which is divisible by 4.

So, $(a, a) \in S \ \forall \ a \in A$

Hence S is reflexive. [½]

(ii) Symmetric :

Let $a, b \in A$ such that $(a, b) \in S$

i.e. $|a-b|$ is divisible by 4.

$\Rightarrow$ $|-(b-a)| = |b-a|$ is divisible by 4.

Hence $(b, a) \in S$.

So, S is symmetric. [1]

(iii) Transitive :

Let $a, b, c \in A$ such that $(a, b), (b, c) \in S$

i.e. $|a-b|$ & $|b-c|$ is divisible by 4.

$\Rightarrow$ $|a-b| = 4k_1$ (say)

& $|b-c| = 4k_2$ (say)

$\Rightarrow$ $(a-b) = \pm 4k_1$ ....(i)

& $(b-c) = \pm 4k_2$ ....(ii)

add equation (i) & (ii)

$\Rightarrow$ $(a - b) + (b - c) = \pm 4k_1 \pm 4k_2$

$\Rightarrow$ $a - c$ is divisible by 4.

$\Rightarrow$ $|a - c|$ is divisible by 4.

Hence, $(a, c) \in S$

So, S is transitive. [2]

Hence S is an equivalence relation.

Further, let $(x, 1) \in S, x \in A$

$\Rightarrow$ $|x - 1|$ is divisible by 4

$\Rightarrow$ $x - 1 = 0, 4 \& 8,$

$\Rightarrow$ $x = 1, 5, 9$

Required set is $\{1, 5, 9\}$. [½]

11. Let $f : R \rightarrow R$ be defined as $f(x) = 10x + 7$. Find the function $g : R \rightarrow R$ such that $gof = fog = I_R$.

[CBSE 2011, 4M]

OR

A binary operation * on the set {0,1,2,3,4,5} is defined as : $a * b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b \geq 6 \end{cases}$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$, being the inverse of 'a'.

Sol. Given $f : R \rightarrow R$, $f(x) = 10x + 7$

and $g : R \rightarrow R$ such that $gof = fog = I_R$

$\Rightarrow gof(x) = fog(x) = I_R(x), \forall x \in R$

i.e. f is invertible function and g is its inverse function [1]

Let $f(x) = y, \ \forall x \in R$

$\Rightarrow$ $f^{-1}(y) = x, \ \forall y \in R$

$\Rightarrow$ $g(y) = x, \ \forall y \in R$ ....(1) [1½]

$\because$ $f(x) = y = 10x + 7$

$$\Rightarrow \quad x = \frac{y-7}{10}$$

$$\Rightarrow \quad g(y) = \frac{y-7}{10}$$

$$\Rightarrow \quad g(x) = \frac{y-7}{10} \ \forall x \in R$$ [1½]

OR

I. Let given set A = {0, 1, 2, 3, 4, 5}

let $e \in A$, be an identity element for * on A

By def. : $a * e = a = e * a \ \forall a, e \in S$

Case-I : If $a + e < 6$

$a * e = a$

$a + e = a$

$e = 0 \in A$

Case-II : $a + e \geq 6$

$a * e = a$

$a + e - 6 = a$

$e = 6 \notin A$

Hence Identity ele. $e = O \in A$ [2]

II. For Inverse element :

let $a^{-1} \in A$ be an inverse element of $a \in A$, then

By def. : $a * a^{-1} = e = a^{-1} * a$.

Case-I : If $a + a^{-1} < 6$

$a * a^{-1} = e$

$a + a^{-1} = 0$

$a^{-1} = -a \notin A$

Case-II : $a + a^{-1} \geq 6$

$a * a^{-1} = e$

$a + a^{-1} - 6 = 0$

$a^{-1} = 6 - a \in A ; a \neq 0$

Hence Inverse ele. $a^{-1} = 6 - a; a \neq 0$ [2]

12. Show that $f : N \to N$, given by

$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$ is both one-one and onto. **[CBSE 2012, 4M]**

OR

Consider the binary operations $* : R \times R \to R$ and $o : R \times R \to R$ defined as $a * b = |a - b|$ and $a \, o \, b = a$ for all $a, b \in R$. Show that '$*$' is commutative but not associative, 'o' is associative but not commutative.

Sol. $f : N \to N$, such that

$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$

For one-one :

Case-I : Let n_1 and n_2 both are even

$x_1, x_2 \in N$.

if $f(x_1) = f(x_2)$

$x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one. [½]

Case-II : Let x_1 and x_2 both are odd

$x_1, x_2 \in N$

if $f(x_1) = f(x_2)$

$x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one [½]

Case-III : Let x_1 = odd and x_2 = even $x_1, x_2 \in N$

if $f(x_1) = f(x_2)$

$x_1 + 1 = x_2 - 1$

$x_2 - x_1 = 2$

This is contradiction,

Since the difference between

an odd and even whole number.

can never be 2.

Thus is this case $f(x_1) \neq f(x_2)$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$\therefore$ f is one-one. [1]

For onto :

Case-I : When x is odd in this case x + 1 is even.

$f(x+1) = x + 1 - 1 = x$ [½]

Case-II : When x is even in this case x – 1 is odd.

$f(x-1) = x - 1 + 1 = x$ [½]

Thus every $x \in N$ has its pre image in N.

So range = codomain [1]

Hence f is onto.

OR

For all $a, b \in R$, $a * b = |a - b|$

$= |-(b-a)| = |b - a| = b * a \ \forall a, b \in R$

$\Rightarrow$ $a * b = b * a$

$\Rightarrow$ ' * ' is commutative [1]

Also, for all $a, b, c \in R$,

$(a * b) * c = |a - b| * c = ||a - b| - c|$

and $a * (b * c) = a * |b - c| = |a - |b - c||$

$\Rightarrow$ $(a * b) * c \neq a * (b * c)$

$(\because \ ||a - b| - c| \neq |a - |b - c||$

$\therefore$ '*' is not associative [1]

Also given binary operation

O : R × R → R defined as

Again aob = a and boa = b

$\Rightarrow$ aob ≠ boa

$\Rightarrow$ O is not commutative. [1]

However, for all $a, b, c \in R$,

(aob) oc = aoc = a

and ao(boc) = aob = a

$\Rightarrow$ (aob) oc = ao (boc) $\forall$ a, b, c $\in$ R

$\Rightarrow$ 'O' is associative. [1]

Hence binary opration O is associative but not commutative

14. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers. **[CBSE 2013, 4M]**

Sol. $f : R_+ \to [4, \infty)$

$f(x) = x^2 + 4$

f is invertible $\Rightarrow$ f is one-one and onto

for one-one : Let $x_1, x_2 \in R_+$

we have $f(x_1) = f(x_2)$

$x_1^2 + 4 = x_2^2 + 4$

$(x_1 + x_2)(x_1 - x_2) = 0$

$x_1 = x_2 \quad \because \quad x_1 + x_2 \neq 0$

$f(x_1) = f(x_2) \Rightarrow \quad x_2 = x_2$

f is one-one [1½]

for onto : Let $y = f(x) : x = f^{-1}(y)$ ....(1)

$y = x^2 + 4$

$x = \sqrt{y-4} \in R_+$ ....(2)

if $\quad y - 4 \geq 0$

$y \geq 4$

range of function = $[4, \infty)$

codomain = $[4, \infty)$

range = codomain [1½]

f is onto

f is one-one and onto $\Rightarrow$ f is invertible [½]

from equation (1) & (2)

$f^{-1}(y) = \sqrt{y-4}$

15. If the function $f : R \to R$ be given by $f(x) = x^2 + 2$ and $g : R \to R$ be given by $g(x) = \frac{x}{x-1}$, $x \neq 1$, find fog and gof and hence find fog(2) and gof(–3). **[CBSE 2014, 4M]**

Sol. $f : R \to R; f(x) = x^2 + 2$

$g : R \to R; g(x) = \frac{x}{x-1}, x \neq 1$

$fog = f(g(x))$

$= f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2$

$= \frac{x}{(x-1)^2} + 2 = \frac{x^2 + 2(x-1)^2}{(x-1)^2}$

$= \frac{x^2 + 2x^2 - 4x + 2}{(x-1)^2} = \frac{3x^2 - 4x + 2}{(x-1)^2}$ [1]

$gof = g(f(x))$

$= (x^2 + 2)$

$= \frac{(x^2+2)}{(x^2+2)-1} = \frac{x^2+2}{x^2+1} = 1 + \frac{1}{x^2+1}$ [1]

$\therefore \; fog(2) = \frac{3(2)^2 - 4(2) + 2}{(2-1)^2} = 6$ [½]

$gof(-3) = 1 + \frac{1}{(-3)^2 + 1} = \frac{11}{10} = 1\frac{1}{10}$ [½]

16. Determine whether the relation R defined on the set $\mathbb{R}$ of all real numbers as R = {(a, b) : a, b ∈ $\mathbb{R}$ and $a - b + \sqrt{3} \in S$, where S is the set of all irrational numbers}, is reflexive, symmetric and transitive. **[CBSE 2015, 6M]**

OR

Let $A = \mathbb{R} \times \mathbb{R}$ and * be the binary operation on A defined by (a, b) * (c, d) = (a + c, b + d). Prove that * is commutative and associative. Find the identity element for * on A. Also write the inverse element of the element (3, –5) in A. **[CBSE 2015, 6M]**

Sol. Given $a R b = a - b + \sqrt{3} \in S \ \forall a, b \in R$

Reflexive : Let a ∈ R such that

$$\text{if } a R a \Rightarrow a - a + \sqrt{3}$$

$$= \sqrt{3} \in S, \forall a \in R$$

∴ R is ref. [1½]

Symmetric : Let a, b ∈ R such that

$$\text{if } a R b \Rightarrow a - b + \sqrt{3} \in S$$

$$\Rightarrow -(b - a + \sqrt{3}) + 2\sqrt{3} \in S$$

$$\Rightarrow b R a$$

∴ R is symm. [1½]

Transitive : Let a, b, c ∈ R such that

$$\text{if } \quad a R b = a - b + \sqrt{3} = K_1 \in S \quad \text{......(i)}$$

$$\text{and } \quad b R c = b - c + \sqrt{3} = K_2 \in S \quad \text{......(ii)}$$ [1½]

from (i) + (ii)

$$a - c + 2\sqrt{3} = K_1 + K_2$$

$$\Rightarrow a - c + \sqrt{3} = K_1 + K_2 - \sqrt{3} \in S$$ [1½]

$$\Rightarrow a R c$$ [1½]

∴ R is transitive.

OR

Given (a, b) * (c, d) = (a + c, b + d), (a, b), (c, d) ∈ A

(i) Commutative : (a, b) * (c, d) = (a + c, b + d)

= (c + a d + b) (by com. rule on add)

= (c, d) * (a, b) [1½]

Hence * is commutative

(ii) Associative : Let (a, b) (c, d) (e, f) ∈ A

we have [(a, b) * (c, d)] * (e, f)

⇒ (a + c b + d) * (e, f)

⇒ (a + c + e, b + d + f)

and (a, b) * [(c, d) * (e, f)]

⇒ (a, b) * (c + e, d + f)

⇒ (a + c + e, b + d + f)

⇒ [(a, b) * (c, d)] * (e, f) = (a, b) * [(c, d) * (e, f)]

Hence * is associative. [1½]

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(iii) Identity element : Let $(e_1, e_2) \in A$ is a identity element

By def : $(a, b) * (e_1, e_2) = (a, b)$

$= (a + e_1, b + e_2) = (a, b)$

$= a + e_1 = a \ \& \ b + e_2 = b$

$e_1 = 0 \qquad e_2 = 0$

$(e_1, e_2) = (0, 0) \in A$

Hence (0,0) is an identity element for * on A. [1½]

(iv) Inverse ele. : Let (x, y) is inverse ele of (3, –5)

By def ; $(3, -5) * (x, y) = (0,0)$

$(3 + x, -5 + y) = (0, 0)$

$3 + x = 0$ and $-5 + y = 0$

$x = -3$ and $y = 5$

Hence (–3, 5) is inverse ele of (3, –5) for * on A [1½]

17. Let $A = R \times R$ and * be a binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that * is commutative and associative. Find the identity element for * on A. Also find the inverse of every element $(a, b) \in A$. **[CBSE 2016, 6M]**

Sol. Given $(a, b) * (c, d) = (a + c, b + d)$, $(a, b), (c, d) \in A$

(i) Commutative : $(a, b) * (c, d) = (a + c, b + d)$

$= (c + a \quad d + b)$ (by com. rule on add)

$= (c, d) * (a, b)$ [1½]

Hence * is commutative

(ii) Associative : Let (a, b) (c, d) (e, f) ∈ A

we have $[(a, b) * (c, d)] * (e, f)$

$\Rightarrow \quad (a + c \quad b + d) * (e, f)$

$\Rightarrow \quad (a + c + e, \ b + d + f)$

and $(a, b) * [(c, d) * (e, f)]$

$\Rightarrow \quad (a, b) * (c + e, d + f)$

$\Rightarrow \quad (a + c + e, b + d + f)$

$\Rightarrow \quad [(a, b) * (c, d)] * (e, f) = (a, b) * [(c, d) * (e, f)]$

Hence * is associative. [1½]

(iii) Identity element : Let $(e_1, e_2) \in A$ is a identity element

By def : $(a, b) * (e_1, e_2) = (a, b)$

$= (a + e_1, b + e_2) = (a, b)$

$= a + e_1 = a \ \& \ b + e_2 = b$

$e_1 = 0 \qquad e_2 = 0$

$(e_1, e_2) = (0, 0) \in A$

Hence (0,0) is an identity element for * on A. [1½]

(iv) Inverse ele. : Let (x, y) is inverse ele of (a, b)

By def : (a, b) * (x, y) = (0,0)

(a + x, b + y) = (0, 0)

a + x = 0 and b + y = 0

x = –a and y = –b

Hence (–a, –b) is inverse ele of (a, b) for * on A **[1½]**