MISCELLANEOUS EXAMPLES

Very Short Answer : [1 Mark]

1. Using principal value, evaluate the following : $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ **[CBSE 2008, 2011 1M]**

Sol. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$$=\cos^{-1}\left[\cos\left(\pi-\frac{\pi}{3}\right)\right]+\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]$$

$$=\pi-\cos^{-1}\left(\cos\frac{\pi}{3}\right)+\frac{\pi}{3}=\pi-\frac{\pi}{3}+\frac{\pi}{3}=\pi$$ **[1]**

$\frac{360}{3}$ 120 $-\frac{\pi}{2},\frac{\pi}{2}$ -90 – 90

2. Write the principal value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ **[CBSE 2009, 1M]**

Sol. $\cos^{-1}\cos\left(2\pi-\frac{5\pi}{6}\right)=\frac{5\pi}{6}$ **[1]**

3. Write the principal value of $\sec^{-1}(-2)$. **[CBSE 2010, 1M]**

Sol. $\sec^{-1}(-2)=\pi-\sec^{-1}(2)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$ **[1]**

4. Write the principal value of $\cot^{-1}\left(-\sqrt{3}\right)$. **[CBSE 2010, 1M]**

Sol. $\cot^{-1}(-\sqrt{3})=\pi-\cot^{-1}(\sqrt{3})=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$ **[1]**

5. Find the value of $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$. **[CBSE 2010, 1M]**

$(\pi-\frac{\pi}{5})$

Sol. $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)\Rightarrow\sin^{-1}\sin\left(\frac{\pi}{5}\right)=\frac{\pi}{5}$ **[1]**

6. Find the principal value of $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)$ **[CBSE 2012, 1M]**

Sol. $\tan^{-1}(\sqrt{3})-\sec^{-1}(-2)$

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right)-\sec^{-1}\left(-\sec\left(\frac{2\pi}{3}\right)\right)=\tan^{-1}\left(\tan\frac{\pi}{3}\right)-\sec^{-1}\left(\sec\left(\pi-\frac{\pi}{3}\right)\right)$$

$$=\frac{\pi}{3}-\left(\pi-\frac{\pi}{3}\right);=\frac{\pi}{3}-\frac{2\pi}{3}=-\frac{\pi}{3}$$

$$\because\ \frac{\pi}{3}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\text{ and }\frac{2\pi}{3}\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$$

$$=\frac{\pi}{3}-\frac{2\pi}{3}=-\frac{\pi}{3}$$ **[1]**

7. Write the principal value of $\tan^{-1}(\sqrt{3})-\cot^{-1}(-\sqrt{3})$. **[CBSE 2013, 1M]**

$\frac{\pi}{3}-\left(\pi-\frac{\pi}{6}\right)$

Sol. $\tan^{-1}(\sqrt{3})-\cot^{-1}(-\sqrt{3})$

$\Rightarrow \quad \tan^{-1}(\sqrt{3})-[\pi-\cot^{-1}\sqrt{3}]$

$\Rightarrow \quad \tan^{-1}\left[\tan\frac{\pi}{3}\right]-\left[\pi-\cot^{-1}\left(\cot\frac{\pi}{6}\right)\right]$

$\frac{\pi}{3}-\pi+\frac{\pi}{6} \Rightarrow \frac{\pi}{2}-\pi=-\frac{\pi}{2}$ [1]

8. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$. **[CBSE 2013, 1M]**

Sol. $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$\tan^{-1}\left[2\sin\left(2\times\frac{\pi}{6}\right)\right]$

$\tan^{-1}\left[2\sin\frac{\pi}{3}\right]$

$\tan^{-1}\left[2\times\frac{\sqrt{3}}{2}\right] \Rightarrow \tan^{-1}\sqrt{3}=\frac{\pi}{3}$ [1]

9. If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$ $xy<1$, then write the value of $x+y+xy$. **[CBSE 2014, 1M]**

Sol. $\tan^{-1}\left[\frac{x+y}{1-xy}\right]=\tan^{-1}(1)$

$x+y=1-xy$

$x+y+xy=1$ [1]

Short Answer : [4 Marks]

10. Prove the following : $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$. **[CBSE 2008,10, 4M]**

Sol. $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$

LHS $=\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)$

$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right.$

where $x>0, y>0, xy<1]$

Cont.

$$=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3\times 5}}\right)+\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7\times 8}}\right)$$ [2]

$$=\tan^{-1}\left(\frac{8}{14}\right)+\tan^{-1}\left(\frac{15}{55}\right)$$

$$=\tan^{-1}\left(\frac{4}{7}\right)+\tan^{-1}\left(\frac{3}{11}\right)$$ [1]

$$=\tan^{-1}\left(\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{12}{77}}\right)=\tan^{-1}\left(\frac{44+21}{77-12}\right)=\tan^{-1}\left(\frac{65}{65}\right)$$

$$=\tan^{-1}(1)=\frac{\pi}{4}=\text{RHS} \quad \text{(Proved)}$$ [1]

11. Prove the following :

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2},\quad x\in\left(0,\frac{\pi}{4}\right)$$

OR

Solve for x : $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$ **[CBSE 2009, 4M]**

Sol. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

Now $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})^2}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})(\sqrt{1+\sin x}+\sqrt{1-\sin x})}$$ [1]

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{1-\sin^2 x}}{(1+\sin x)-(1-\sin x)}$$ [1]

$$=\frac{2+2\cos x}{2\sin x}=\frac{1+\cos x}{\sin x}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\cot\frac{x}{2}$$ [1]

$$\therefore\ \cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}.$$ [1]

OR

$2\tan^{-1}(\cos x) = 2\ \text{cosec}\ x$

$\Rightarrow \dfrac{2\cos x}{1-\cos^2 x} = \dfrac{2}{\sin x} \Rightarrow 2\sin x\cos x = 2\sin^2 x$ [2]

$\Rightarrow \sin x\cos x - \sin^2 x = 0$

$\Rightarrow \sin x(\cos x - \sin x) = 0$ [1]

$\Rightarrow \sin x = 0$ and $\tan x = 1$ [1]

(Not possible) and $x = n\pi + \pi/4$

12. Prove the following : $\tan^{-1} x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$ **[CBSE 2010, 4M]**

OR

Prove the following : $\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1} x\right)\right\}\right] = \sqrt{\dfrac{1+x^2}{2+x^2}}$.

Sol. L.H.S. $= \tan^{-1} x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

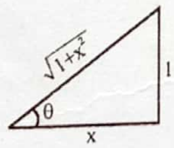
$$= \tan^{-1}\dfrac{\left(x + \dfrac{2x}{1-x^2}\right)}{1 - x\dfrac{(2x)}{1-x^2}}$$ [2]

$$= \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$$ [2]

OR

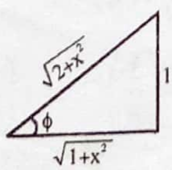
Let $\cot^{-1} x = \theta$

$\Rightarrow \cot\theta = x$



$\Rightarrow \sin\theta = \dfrac{1}{\sqrt{1+x^2}}$

$\cos\left(\tan^{-1}\dfrac{1}{\sqrt{1+x^2}}\right)$ [2]



again let $\tan^{-1}\dfrac{1}{\sqrt{1+x^2}} = \phi$

$\Rightarrow \tan\phi = \dfrac{1}{\sqrt{1+x^2}} \Rightarrow \cos\phi = \sqrt{\dfrac{1+x^2}{2+x^2}}$ [2]

PBP HMB

13. Solve for x : $\tan^{-1}\left(\dfrac{x-1}{x-2}\right) + \tan^{-1}\left(\dfrac{x+1}{x+2}\right) = \dfrac{\pi}{4}$. **[CBSE 2010, 4M]**

Sol. $\tan^{-1}\left(\dfrac{x-1}{x-2}\right)+\tan^{-1}\left(\dfrac{x+1}{x+2}\right)=\dfrac{\pi}{4}$

$$LHS=\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)$$

$$=\tan^{-1}\left[\frac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\left(\dfrac{x-1}{x-2}\right)\left(\dfrac{x+1}{x+2}\right)}\right]$$ [2]

$$=\tan^{-1}\left[\frac{(x-1)(x+2)+(x-2)(x+1)}{(x^2-4)-(x^2-1)}\right]$$

$$=\tan^{-1}\left[\frac{x^2+2x-x-2+x^2-2x+x-2}{-3}\right]$$

$$=\tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$ [1]

$\Rightarrow$ taking 'tan' on both the sides $\dfrac{2x^2-4}{-3}=1$

$\Rightarrow$ $2x^2-4+3=0$ $\quad\Rightarrow\quad$ $2x^2-1=0$

$\Rightarrow$ $x=\pm\dfrac{1}{\sqrt{2}}$ [1]

14. Prove the following : $2\tan^{-1}\left(\dfrac{1}{2}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)=\tan^{-1}\left(\dfrac{31}{17}\right)$. **[CBSE 2011, 4M]**

Sol. We have, $2\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{7}$

$$=\tan^{-1}\left\{\frac{2\times\dfrac{1}{2}}{1-\left(\dfrac{1}{2}\right)^2}\right\}+\tan^{-1}\frac{1}{7} \qquad \left[\because\ 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right),\text{if } -1<x<1\right]$$ [2]

$$=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}$$ [1]

$$=\tan^{-1}\left\{\frac{\dfrac{4}{3}+\dfrac{1}{7}}{1-\dfrac{4}{3}\times\dfrac{1}{7}}\right\}=\tan^{-1}\frac{31}{17}$$ [1]

15. Prove the following : $\cos\left(\sin^{-1}\dfrac{3}{5}+\cot^{-1}\dfrac{3}{2}\right)=\dfrac{6}{5\sqrt{13}}$ **[CBSE 2012, 4M]**

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Sol. LHS $= \cos\left[\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right]$ $= \frac{6}{5\sqrt{13}}$

$$= \cos\left[\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right] \quad [1]$$

$$= \cos\left[\cos^{-1}\left\{\frac{4}{5}\times\frac{3}{\sqrt{13}}-\sqrt{1-\left(\frac{4}{5}\right)^2}\times\sqrt{1-\left(\frac{3}{\sqrt{13}}\right)^2}\right\}\right] \quad [1]$$

$$\left[\because \cos^{-1}x+\cos^{-1}y=\cos^{-1}\left[xy-\sqrt{1-x^2}\sqrt{1-y^2}\right]\right]$$

$$= \frac{12}{5\sqrt{3}}-\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{9}{13}} \quad [1]$$

$$= \frac{12}{5\sqrt{3}}-\frac{3}{5}\times\frac{2}{\sqrt{13}}=\frac{6}{5\sqrt{13}}=\text{RHS} \quad [1]$$

16. Show that : $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$ **[CBSE 2013, 4M]**

OR

Solve the following equation : $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$

Sol. L.H.S. $=\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$

Let $\sin^{-1}\frac{3}{4}=\theta$ then $\sin\theta=\frac{3}{4}$ [1]

$$\cos\theta=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4} \quad [1]$$

$$\text{Now } \tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\sqrt{\frac{1-\sqrt{7}/4}{1+\sqrt{7}/4}}=\frac{\sqrt{4-\sqrt{7}}}{\sqrt{4+\sqrt{7}}}\times\frac{\sqrt{4-\sqrt{7}}}{\sqrt{4-\sqrt{7}}}=\frac{4-\sqrt{7}}{3} \quad [2]$$

OR

$$\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$$

$$\cos\left(\tan^{-1}x\right)=\sin\left(\frac{\pi}{2}-\tan^{-1}\frac{3}{4}\right) \quad [1]$$

$$\cos\left(\tan^{-1}x\right)=\cos\left(\tan^{-1}\frac{3}{4}\right) \quad [1]$$

$$\tan^{-1}x=\tan^{-1}\frac{3}{4} \quad [1]$$

$$x=\frac{3}{4} \quad [1]$$

17. Prove that $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ \frac{-1}{\sqrt{2}}\le x\le 1$

OR

If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, find the value of x. **[CBSE 2014, 4M]**

Sol. $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ \frac{-1}{\sqrt{2}}\le x\le 1$

In LHS

put $x = \cos 2\theta$

$$\tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$ [½]

$$=\tan^{-1}\left[\frac{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]$$ [1]

$$=\tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$=\tan^{-1}\left[\frac{\tan(\pi/4)-\tan\theta}{1+\tan(\pi/4).\tan\theta}\right]$$

$$=\tan^{-1}[\tan(\pi/4-\theta)]$$ [1½]

$$=\frac{\pi}{4}-\theta \qquad \text{as}\left\{\begin{array}{l} x=\cos 2\theta \\ \text{so, } \theta=\dfrac{\cos^{-1}x}{2}\end{array}\right\}$$

$$=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = \text{RHS} \qquad \text{proved}$$ [1]

OR

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4} \qquad ...(i)$$

Use formula, $\tan^{-1}\left[\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\left(\frac{x-2}{x-4}\right)\left(\frac{x+2}{x+4}\right)}\right]=\frac{\pi}{4}$ [1½]

$$\Rightarrow \tan^{-1}\left[\frac{(x-2)(x+4)+(x+2).(x-4)}{(x-4).(x+4)-(x-2).(x+2)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \frac{(x-2)(x+4)+(x+2).(x-4)}{(x-4).(x+4)-(x-2).(x+2)}=1$$

$$\Rightarrow \frac{x^2-8-2x+x^2-8+2x}{x^2-16-x^2+4}=1$$ [1½]

$$\Rightarrow \frac{2x^2-16}{-12}=1$$

$\Rightarrow$ $2x^2 = -12 + 16 = 4$

$\Rightarrow$ $x^2 = 2$ $\quad \Rightarrow \quad x = \pm\sqrt{2}$ [1]

18. Evaluate : $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)+\frac{\pi}{4}\right\}$ **[CBSE 2015, 4M]**

$2\tan^{-1}x = \left(\frac{2x}{1+x^2}\right)$

Sol. $\tan\left\{2\tan^{-1}\frac{1}{5}+\frac{\pi}{4}\right\}$

$$\tan\left[\tan^{-1}\left(\frac{2\sqrt{5}}{1-\frac{1}{25}}\right)+\frac{\pi}{4}\right]$$ [½]

$$\Rightarrow \tan\left[\frac{\pi}{4}+\tan^{-1}\left(\frac{5}{12}\right)\right]$$ [½]

$$\Rightarrow \frac{1+\tan\left(\tan^{-1}\frac{5}{12}\right)}{1-\tan\left(\tan^{-1}\frac{5}{12}\right)}$$

$$\Rightarrow \frac{1+\frac{5}{12}}{1-\frac{5}{12}}=\frac{17}{7}$$ [1½]

19. Solve for x : $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$. **[CBSE 2016, 4M]**

OR

Prove that $\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right)-\tan^{-1}\left(\frac{4x}{1-4x^2}\right)=\tan^{-1}2x; |2x|<\frac{1}{\sqrt{3}}$

Sol. $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$

$\Rightarrow$ $\tan^{-1}(x-1)+\tan^{-1}(x+1)=\tan^{-1}3x-\tan^{-1}x.$ [½]

$$\Rightarrow \tan^{-1}\left(\frac{2x}{2-x^2}\right)=\tan^{-1}\left(\frac{2x}{1+3x^2}\right)$$ [1½]

$\Rightarrow$ $2x(1+3x^2)=(2-x^2)2x$ [½]

$\Rightarrow$ $2x[4x^2-1]=0$ [½]

$\Rightarrow$ $x=0$ and $x=\pm\frac{1}{2}$ [1]

OR

$$\text{LHS} = \tan^{-1}\left[\frac{3(2x)-(2x)^3}{1-3(2x)^2}\right] - \tan^{-1}\left[\frac{2.(2x)}{1-(2x)^2}\right]$$

put $2x = \tan\theta;\ \theta = \tan^{-1}2x.$ **[1]**

$$\tan^{-1}\left[\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right] - \tan^{-1}\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]$$ **[1]**

$$\tan^{-1}(\tan 3\theta) - \tan^{-1}(\tan 2\theta)$$ **[1]**

$3\theta - 2\theta$

$\theta = \tan^{-1}2x = \text{RHS}.$ **[1]**