MISCELLANEOUS EXAMPLES

Very Short Answer : [1 Mark]

[CBSE 2007, 1M]

1. Evaluate : $\int x \log 2x dx$.

Sol. $I = \int x \log 2x \, dx$

$$I = \log 2x \int x \, dx - \int \left[\frac{d}{dx} \log 2x \int x \, dx \right] dx + c$$

$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{1}{2x} . 2 . \frac{x^2}{2} dx + c$$

$$= \frac{x^2}{2} \log 2x - \frac{1}{2} . \frac{x^2}{2} + c = \frac{x^2}{2} \log 2x - \frac{x^2}{4} + c$$ [1]

[CBSE 2007, 1M]

2. Evaluate : $\int \sin 7x \sin x dx$

Sol. Let $I = \int \sin 7x \sin x dx = \frac{1}{2} \int 2 \sin 7x \sin x dx$

$$= \frac{1}{2} \int [\cos(7x - x) - \cos(7x + x)] dx$$

$$= \frac{1}{2} \int [\cos 6x - \cos 8x) dx = \frac{1}{2} \left[\frac{\sin 6x}{6} - \frac{\sin 8x}{8} \right] + c$$

$$= \frac{\sin 6x}{12} - \frac{\sin 8x}{16} + c$$ [1]

[CBSE 2008, 1M]

3. Evaluate : $\int \frac{2 \cos x}{3 \sin^2 x} dx$.

Sol. Let $I = \int \frac{2 \cos x}{3 \sin^2 x} dx = \frac{2}{3} \int \frac{1}{\sin x} . \frac{\cos x}{\sin x} dx$

$$= \frac{2}{3} \int \operatorname{cosec} x \cot x dx = -\frac{2}{3} \operatorname{cosec} x + c$$ [1]

[CBSE 2008, 1M]

4. Evaluate : $\int_0^1 \frac{2x}{1 + x^2} dx$.

Sol. Let $I = \int_0^1 \frac{2x}{1 + x^2} dx$

Put $1 + x^2 = t$

$2x \, dx = dt$

when $x = 0 \Rightarrow t = 1$

$x = 1 \Rightarrow t = 2$

$$\therefore \ I = \int_1^2 \frac{dt}{t} = [\log |t|]_1^2 = \log 2 - \log 1 = \log 2$$ [1]

CBSE

5. Evaluate : $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ **[CBSE 2009, 1M]**

$\frac{1}{2\sqrt{x}}dx = dt$

$\frac{1}{\sqrt{x}}dx = 2dt$

Sol. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

Let $\sqrt{x} = t \quad \Rightarrow \quad \frac{1}{2\sqrt{x}}dx = dt$

Now $\int 2\cos t\,dt = 2\sin t + C = 2\sin\sqrt{x} + C$ **[1]**

6. Evaluate : $\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}dx$ **[CBSE 2009, 1M]**

Sol. $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}dx = |\sin^{-1}x|_0^{1/\sqrt{2}} = \frac{\pi}{4}$ **[1]**

7. Evaluate : $\int \sec^2(7-4x)dx$ **[CBSE 2010, 1M]**

Sol. $I = \int \sec^2(7-4x)dx$

put $7-4x = t \quad \Rightarrow -4dx = dt \Rightarrow dx = \frac{-1}{4}dt$

$I = -\frac{1}{4}\int \sec^2 t\,dt$

$= -\frac{1}{4}\tan t + C = -\frac{1}{4}\tan(7-4x) + C$ **[1]**

8. Write the value of the following integral : $\int_{-\pi/2}^{\pi/2} \sin^5 x\,dx$ **[CBSE 2010, 1M]**

Sol. $I = \int_{-\pi/2}^{\pi/2} \sin^5 x\,dx = 0 \quad \left[\int_{-a}^{a} f(x)dx = 0, \text{ if } f(x) \text{ is an odd function}\right]$ **[1]**

9. Evaluate : $\int \frac{dx}{\sqrt{1-x^2}}$ **[CBSE 2011, 1M]**

Sol. $\int \frac{dx}{\sqrt{1-x^2}}$

Put $x = \sin\theta \quad \Rightarrow \quad dx = \cos\theta d\theta$

$\int \frac{\cos\theta d\theta}{\cos\theta} = \theta + c = \sin^{-1}x + c$ **[1]**

10. Evaluate : $\int \frac{(\log x)^2}{x}dx$ **[CBSE 2011, 1M]**

Sol. Let $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$$\int t^2 dt = \frac{t^3}{3} + c = \frac{(\log x)^3}{3} + c$$ [1]

11. Evaluate : $\int_0^2 \sqrt{4-x^2}$ **[CBSE 2012, 1M]**

Sol. $\int_0^2 \sqrt{4-x^2}\,dx = \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 2 \times \frac{\pi}{2} - 0 = \pi$ [1]

12. Evaluate : $\int_2^4 \frac{x}{x^2+1}dx$ **[CBSE 2014, 1M]**

Sol. $I = \int_2^4 \frac{x}{x^2+1}dx$

put $x^2 + 1 = t \Rightarrow 2xdx = dt$ | at $x = 2$

$xdx = \frac{1}{2}dt$ | $t = 5$

at $x = 4$

$t = 17$

$\therefore \quad I = \int_5^{17} \frac{1/2}{t}dt$

$= \frac{1}{2}[\log|t|]_5^{17}$

$= \frac{1}{2}[\log 17 - \log 5]$

$= \frac{1}{2}\log(17/5)$ [1]

13. Evaluate : $\int_e^{e^2} \frac{dx}{x\log x}$ **[CBSE 2014, 1M]**

logx = t

$\frac{1}{x}dx = dt$

$\int \frac{1}{t}$

(log logx)

Sol. $I = \int_e^{e^2} \frac{dx}{x\log x}$

$= \left[\log(\log x)\right]_e^{e^2}$

$= \log(\log e^2) - \log(\log e)$

$= \log 2$ [1]

14. If $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$, find the value of a. **[CBSE 2014, 1M]**

Sol. $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$

$$\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a = \frac{\pi}{8}$$

$$\frac{1}{2}\tan^{-1}\frac{a}{2} = \frac{\pi}{8}$$

$$\frac{a}{2} = \tan\frac{\pi}{4}$$

$$a = 2$$ **[1]**

15. If $f(x) = \int_0^x t\sin t\, dt$, then write the value of $f'(x)$. **[CBSE 2014, 1M]**

Sol. $\frac{d}{dx}[f(x)] = \frac{d}{dx}\int_0^x t\sin t dt$

$f'(x) = [t\sin t]_0^x = x\sin x$ **[1]**

Short Answer : [4 Marks]

16. Evaluate : $\int \frac{x+2}{2x^2+6x+5}dx$. **[CBSE 2007, 4M]**

Sol. Let $I = \int \frac{x+2}{2x^2+6x+5}dx$

Put $2x^2 + 6x + 5 = t$

$(4x+6)dx = dt$ **[½]**

$$I = \frac{1}{4}\int \frac{4x+8}{2x^2+6x+5}dx$$

$$I = \frac{1}{4}\int \frac{4x+6+2}{2x^2+6x+5}dx$$

$$I = \frac{1}{4}\int\left(\frac{4x+6}{2x^2+6x+5} + \frac{2}{2x^2+6x+5}\right)dx$$ **[1]**

$$I = \frac{1}{4}\int \frac{4x+6}{2x^2+6x+5}dx + \frac{1}{4}\int \frac{2}{2\left(x^2+3x+\frac{5}{2}\right)}dx$$

$$I = \frac{1}{4}\int\frac{dt}{t} + \frac{1}{4}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + \frac{5}{2}}$$ **[1]**

$x+2 = A\frac{d}{dx}(2x^2+6x+5) + B$

$x+2 = A(4x+6) + B$

$1 = 4A$

$A = \frac{1}{4}$

$$I=\frac{1}{4}\log|t|+\frac{1}{4}\int\frac{dx}{\left(x+\frac{3}{2}\right)^2+\frac{1}{4}}+c$$

$$I=\frac{1}{4}\log|2x^2+6x+5|+\frac{1}{4.\frac{1}{2}}\tan^{-1}\left(\frac{x+\frac{3}{2}}{\frac{1}{2}}\right)+c$$ [1]

$$I=\frac{1}{4}\log|2x^2+6x+5|+2\tan^{-1}(2x+3)+c$$ [½]

17. Evaluate : $\int\frac{(2x+1)}{(x+2)(x-3)}dx$ **[CBSE 2007, 4M]**

Sol. Let $I=\int\frac{2x+1}{(x+2)(x-3)}dx$

Let $\frac{2x+1}{(x+2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-3}$ [½]

$$\frac{2x+1}{(x+2)(x-3)}=\frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$$

$2x+1=A(x-3)+B(x+2)$

Put $x=3$, we get

$6+1=0+B(3+2)\Rightarrow B=\frac{7}{5}$ [1]

Put $x=-2$, we get,

$-4+1=A(-2,-3)+0\Rightarrow A=\frac{3}{5}$ [1]

$$I=\int\frac{\frac{3}{5}}{x+2}dx+\int\frac{\frac{7}{5}}{x-3}dx$$

$$=\frac{3}{5}\log|x+2|+\frac{7}{5}\log|x-3|+c$$ [1½]

18. Evaluate : $\int\frac{x}{x^2+x+1}dx$ **[CBSE 2007, 4M]**

Sol. $I=\int\frac{x}{x^2+x+1}dx$

Put $x^2 + x + 1 = t$

$(2x + 1)\,dx = dt$ [½]

$$I = \frac{1}{2}\int \frac{2x}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1-1}{x^2+x+1}dx$$

$x = A\frac{d}{dx}(x^2+x+1) + B$

$x = (2x+1)A + B$

$1 = 2A$

$A = 1/2$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$ [1]

$$= \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1}$$

$$= \frac{1}{2}\log|t| - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$ [1]

$$= \frac{1}{2}\log|x^2+x+1| - \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$ [1]

$$= \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$ [½]

2 times

19. Using properties of definite integrals, evaluate $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$. **[CBSE 2007, 2011 4M]**

Sol. Let $I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$

$f(a-x)$

$$I = \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx \quad \text{[By property]}$$ [1]

$$I = \int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$ [1]

$$I = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$I = \int_0^{\frac{\pi}{4}} \log 2\,dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx = \log 2\,[x]_0^{\frac{\pi}{4}} - I$$ [1]

$$I + I = \log 2\left[\frac{\pi}{4} - 0\right]$$

$$2I = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2.$$ [1]

20. Evaluate : $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ **[CBSE 2008, 2010, 4M]**

Sol. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ ...(i)

By prop. $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx$$ ...(ii) [1]

On adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx$$

$$2I = \pi\int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{(\sec^2 x - \tan^2 x)} dx$$ [1]

$$= \pi\left[\int_0^{\pi} \sec x \tan x dx - \int_0^{\pi} \tan^2 x dx\right]$$

$$= \pi\left[(\sec x)_0^{\pi} - \int_0^{\pi} (\sec^2 x - 1)\,dx\right]$$ [1]

$$= \pi\left[(-2) - (\tan x - x)_0^{\pi}\right]$$

$$2I = \pi(\pi - 2)$$

$$I = \pi\left(\frac{\pi}{2} - 1\right)$$ [1]

d/dx tan = sec²x
d/dx sec x = sec x tan x.

21. Evaluate : $\int_0^{\frac{\pi}{2}} \log \sin x dx.$ **[CBSE 2008, 4M]**

Sol. Let $I = \int_0^{\pi/2} \log \sin x \, dx$ ...(i)

$I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$I = \int_0^{\pi/2} \log \cos x \, dx$ ...(ii) [1]

On adding (i) and (ii), we get

$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) dx$

$2I = \int_0^{\pi/2} \log \sin x \cos x \, dx$

$2I = \int_0^{\pi/2} \log \frac{\sin 2x}{2} dx$ $[\because \sin 2x = 2 \sin x \cos x]$ [½]

$2I = \int_0^{\pi/2} (\log \sin 2x - \log 2) dx$

$2I = \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx$ [½]

Put $2x = t \Rightarrow 2\,dx = dt$

when $x = 0 \quad \therefore \quad t = 0$

when $x = \frac{\pi}{2} \quad \therefore \quad t = 2 \cdot \frac{\pi}{2} = \pi$ [½]

$\therefore \quad 2I = \int_0^{\pi} \log \sin t \frac{dt}{2} - \log 2 [x]_0^{\pi/2}$ $\left[\int_0^{\pi} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a - x) = f(x)\right]$

$\because \quad \log \sin(\pi - t) = \log \sin t$

$\therefore \quad 2I = \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log \sin t \, dt - \log 2 \left[\frac{\pi}{2} - 0\right]$ [½]

$2I = \int_0^{\pi/2} \log \sin t \, dt - \frac{\pi}{2} \log 2$

$2I = \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2} \log 2$ $\left[\int_a^b f(x)\,dx = \int_a^b f(x)\,dt\right]$

$2I = I - \frac{\pi}{2} \log 2$

$\therefore \quad I = -\frac{\pi}{2} \log 2.$ [1]

22. Evaluate : $\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$ **[CBSE 2009, 4M]**

Sol. $\int \frac{dx}{\sqrt{5-4x-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$ [1]

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{5}{2}+1-(1+2x+x^2)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{7}{2}-(1+x)^2}}$ [1½]

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(1+x)^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{(1+x)}{\sqrt{\frac{7}{2}}}+C$ [1½]

23. Evaluate : $\int x\sin^{-1}x\,dx$ **[CBSE 2009, 4M]**

Sol. $I = \int \underset{II}{x}\,\underset{I}{\sin^{-1}x}\,dx = \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{x^2dx}{\sqrt{1-x^2}}$ [1]

ILATE

Put $x=\sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{\sin^2\theta\cos\theta d\theta}{\cos\theta}$ [1]

$\cos 2\theta = 1-2\sin^2\theta$

$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{4}\int(1-\cos 2\theta)d\theta$ [1]

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)$

$\frac{\sin 2\theta}{8}$

$= \frac{x^2\sin^{-1}x}{2} - \frac{\sin^{-1}x}{4} + \frac{x\sqrt{1-x^2}}{4}+c$ [1]

24. Evaluate : $\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx$. **[CBSE 2010, 4M]**

Sol. $\int\frac{(5x+3)}{\sqrt{x^2+4x+10}}dx$

$= \int\frac{\left[\frac{5}{2}(2x+4)-7\right]}{\sqrt{x^2+4x+10}}dx$ [1]

$= \frac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}} - 7\int\frac{dx}{\sqrt{(x+2)^2+6}}$ [1]

$= 5\sqrt{x^2+4x+10} - 7\log\left|x+2+\sqrt{x^2+4x+10}\right|+c$ [2]

25. Evaluate the following : $\int_1^2\frac{5x^2}{x^2+4x+3}dx$ **[CBSE 2010, 4M]**

Sol. $\displaystyle\int_1^2 \frac{5x^2}{x^2+4x+3} = \int_1^2 \frac{[5(x^2+4x+3)-20x-15]}{(x^2+4x+3)}dx$ **[1]**

$8x^2 = A\frac{d}{dx}(x^2+4x+3)+B$

$8x^2 = A(2x+4)+B$

$$= 5\int_1^2 dx - 10\int_1^2 \frac{(2x+4)dx}{(x^2+4x+3)} + 25\int_1^2 \frac{dx}{(x+2)^2-1}$$ **[1]**

$$= 5[x]_1^2 - 10\left[\ell n\left|x^2+4x+3\right|\right]_1^2 + \frac{25}{2}\left[\ell n\left|\frac{x+1}{x+3}\right|\right]_1^2$$ **[1]**

$$= 5 - 10\left[\ell n\left|\frac{15}{8}\right|\right] + \frac{25}{2}\ell n\left|\frac{6}{5}\right|$$ **[1]**

26. Evaluate : $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ **[CBSE 2011, 4M]**

Sol. $\displaystyle I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$\frac{\pi}{6}+\frac{\pi}{3} = \frac{3\pi}{6} = \frac{\pi}{2}$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad ....(1)$$ **[1]**

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad ....(2)$$ $\left[\text{Using } \int_a^b f(x) = \int_a^b f(a+b-x)\right]$ **[1]**

add (1) & (2)

$$2I = \int_{\pi/6}^{\pi/3} dx, \Rightarrow 2I = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$$ **[2]**

27. Evaluate : $\displaystyle\int_{-1}^{2} |x^3-x|dx$ **[CBSE 2012, 4M]**

Sol. We note that $x^3 - x \geq 0$ on $[-1, 0]$ and $x^3 - x \leq 0$ on $[0, 1]$ and that $x^3 - x \geq 0$ on $[1, 2]$. So by property of definite integral we get

$$\int_{-1}^{2}|x^3-x|dx = \int_{-1}^{0}(x^3-x)dx + \int_0^1 -(x^3-x)dx + \int_1^2 (x^3-x)dx$$ **[1]**

$$= \int_{-1}^{0}(x^3-x)dx + \int_0^1 (x-x^3)dx + \int_1^2 (x^3-x)dx = \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2$$ **[1]**

$$= -\left(\frac{1}{4}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + (4-2) - \left(\frac{1}{4}-\frac{1}{2}\right) = -\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+2-\frac{1}{4}+\frac{1}{2} = \frac{3}{2}-\frac{3}{4}+2 = \frac{11}{4}$$ **[2]**

28. Evaluate : $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ **[CBSE 2012, 4M]**

Sol. Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos x} dx$. Then, by property

$\because \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)dx}{1+\cos^2(\pi - x)} = \int_0^{\pi} \frac{(\pi - x)\sin x\, dx}{1+\cos^2(\pi - x)} = \pi \int_0^{\pi} \frac{\sin x dx}{1+\cos^2 x} - I \quad [1]$$

or $2I = \pi \int_0^{\pi} \frac{\sin x dx}{1+\cos^2 x}$ or $I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x\, dx}{1+\cos^2 x}$ [1]

Put $\cos x = t$ so that $-\sin x\, dx = dt$. When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$.

$$I = \frac{-\pi}{2} \int_1^{-1} \frac{dt}{1+t^2} = +\frac{\pi}{2} \int_{-1}^{1} \frac{dt}{1+t^2}$$ [By property $\int_a^b f(x)dx = -\int_b^a f(x)dx$] [1]

$$= \pi \int_0^1 \frac{dt}{1+t^2}$$ (since $\frac{1}{1+t^2}$ is even function)

$$= \pi[\tan^{-1} t]_0^1 = \pi[\tan^{-1} 1 - \tan^{-1} 0] = \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4} \quad [1]$$

29. Evaluate : $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$ **[CBSE 2013, 4M]**

Sol. Let $I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

$$\Rightarrow \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx$$ $= 2\cos^2 x - 2\cos^2 \alpha$ [1]

$2(\cos^2 x - \cos^2 \alpha)$

$2(\cos x - \cos \alpha)(\cos x + \cos \alpha)$

$$\Rightarrow 2\int (\cos x + \cos \alpha)dx \quad [2]$$

$$= 2[\sin x + x\cos \alpha] + C \quad [1]$$

30. Evaluate : $\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$ **[CBSE 2013, 4M]**

Sol. $I = \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}} dx$ [1]

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{(x+1)^2 + (\sqrt{2})^2}} \quad [½ + ½]$$

$$= \sqrt{x^2+2x+3} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C. \quad [1 + 1]$$

31. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$ **[CBSE 2013, 4M]**

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Sol. $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$ ....(i)

$I = \int_0^{2\pi} \frac{1}{1+e^{-\sin x}}$ (By $\int_0^a f(x)dx = \int_0^a f(a-x)dx$) [1]

$= \int_0^{2\pi} \frac{e^{\sin x}}{1+e^{\sin x}} dx$ ......(ii) [1]

from (i) + (ii)

$2I = \int_0^{2\pi} \left[\frac{1+e^{\sin x}}{1+e^{\sin x}}\right] dx$ [1]

$2I = \int_0^{2\pi} dx \Rightarrow 2I = (x)_0^{2\pi}$ [½]

$2I = 2\pi \Rightarrow I = \pi$ [½]

$1+\frac{1}{x^3} = t$

$-\frac{3}{x^4}dx = dt$

$\int \frac{dx}{x^4\left(1+\frac{1}{x^3}\right)}$ $\frac{dx}{x^4} = -\frac{1}{3}dt$

32. Evaluate : $\int \frac{dx}{x(x^3+1)}$ **[CBSE 2013, 4M]**

$= -\frac{1}{3}\int \frac{dt}{t}$

$\log\left(1+\frac{1}{x^3}\right)$

$-\frac{1}{3}\log x^3+1 - \log$

Sol. $I = \int \frac{1}{x(x^3+1)} dx$

$\Rightarrow I = \int \frac{x^2}{x^3(x^3+1)} dx$ [1]

Put $\left.\begin{aligned} x^3 &= t \\ 3x^2 dx &= dt \\ x^2 dx &= \frac{dt}{3} \end{aligned}\right\}$

$\Rightarrow I = \frac{1}{3}\int \frac{dt}{t(t+1)} = \frac{1}{3}\int \frac{(t+1)-(t)}{t(t+1)}$ [½]

$\Rightarrow I = \frac{1}{3}\int \left[\frac{1}{t} - \frac{1}{t+1}\right] dt$ [½]

$\Rightarrow I = \frac{1}{3}[\log t - \log t+1] + C$ [1]

$\Rightarrow I = \frac{1}{3}\log \frac{t}{t+1} + C$

$\Rightarrow I = \frac{1}{3}\log \frac{x^3}{x^3+1} + C$ [1]

33. Evaluate : . **[CBSE 2014, 4M]**

$$\int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x}\,dx$$

OR

Evaluate :

$$\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$$

Sol. $I = \int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x}\,dx$

$I = \int_0^{\pi} \frac{4(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$ $\{$Applying $\int f(a-x) = \int f(x)$

$I = \int_0^{\pi} \frac{4\pi \sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x}\,dx$

$I = \int_0^{\pi} \frac{4\pi \sin x}{1+\cos^2 x}\,dx - I$

$2I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$ $\frac{\sin x}{1+\cos^2(x)}$ [1]

$2I = 4\pi . 2 \times \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$ $\left\{\text{Applying } \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x) = f(x)\right.$

$I = 4\pi \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$ [1]

put $\cos x = t$ $\Rightarrow$ $-\sin x\,dx = dt$

as well for $x = 0$, $x = \pi/2$

$t = 1$ $t = 0$

$\therefore$ $I = 4\pi \int_1^{0} \frac{-dt}{1+t^2}\,dt$

$I = 4\pi \int_0^{1} \frac{dt}{1+t^2}\,dt$ $\left\{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right.$ [1]

$I = 4\pi[\tan^{-1} t]_0^1$

$= 4\pi[\tan^{-1} 1 - \tan^{-1} 0]$

$= 4\pi \times \frac{\pi}{4} = \pi^2$ [1]

OR

$$I = \int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$$

put, $x + 2 = \lambda\left(\frac{d}{dx}(x^2+5x+6)\right) + \mu$

$x + 2 = 2\lambda x + 5\lambda + \mu$

comparing coefficients of x both sides

$1 = 2\lambda \quad \Rightarrow \lambda = 1/2$

comparing constant terms both sides,

$2 = 5\lambda + \mu$

or, $2 = 5\left(\frac{1}{2}\right) + \mu$

or, $\mu = 2 - \frac{5}{2} = \frac{-1}{2}$ **[1]**

$$\therefore \int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}\,dx \qquad \{\text{as } x + 2 = \lambda(2x+5) + \mu\}$$

$$\therefore \quad I = \underbrace{\int \frac{\frac{1}{2}(2x+5)}{\sqrt{x^2+5x+6}}\,dx}_{(I_1)} - \underbrace{\frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}}_{(I_2)}$$

$\therefore \quad I = I_1 - I_2 \qquad \ldots(1)$

$I_1 = \frac{1}{2}\int \frac{(2x+5)\,dx}{\sqrt{x^2+5x+6}}$, put $x^2 + 5x + 6 = t$

$\therefore (2x + 5)\,dx = dt$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\left(\frac{t^{-1/2+1}}{-\frac{1}{2}+1}\right) + C = t^{1/2} + C = \sqrt{t} + C = \sqrt{x^2+5x+6} + C$$ **[1]**

$$I_2 = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= \frac{1}{2}.\log\left[\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right] + C$$ **[1]**

Substituting the values of I_1 and I_2 in (1), we get,

$$I = \sqrt{x^2+5x+6} - \frac{1}{2}.\log\left[\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right] + C.$$ **[1]**

34. Evaluate : $\int \frac{\sin x - x\cos x}{x(x+\sin x)}dx$ **[CBSE 2015, 4M]**

OR

Evaluate : $\int \frac{x^3}{(x-1)(x^2+1)}dx$ **[CBSE 2015, 4M]**

Sol. Let $I = \int \frac{\sin x - x\cos x}{x(x+\sin x)}dx$

$$I = \int\left[\frac{\sin x}{x(x+\sin x)} - \frac{x\cos x}{x(x+\sin x)}\right]dx$$ [½]

$$I = \int\left[\frac{(x+\sin x)-x}{x(x+\sin x)} - \frac{\cos x}{x+\sin x}\right]dx$$ [1]

$$I = \int\left[\frac{1}{x} - \frac{1}{x+\sin x} - \frac{\cos x}{x+\sin x}\right]dx$$

$$I = \int \frac{1}{x}dx - \int \frac{1+\cos x}{x+\sin x}dx$$ [1]

by prop $\int \frac{f'(x)}{f(x)}dx = \log f(x) + C$

$$I = \log x - \log(x+\sin x) + C$$ [1]

$$I = \log\frac{x}{x+\sin x} + C$$ [½]

OR

Let $I = \int \frac{x^3}{(x-1)(x^2+1)}dx$

$$\frac{x^3}{(x-1)(x^2+1)} = 1 + \frac{x^2-x+1}{(x-1)(x^2+1)}$$ [½]

Let $\frac{x^2-x+1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+1}$

$x^2 - x + 1 = A(x^2+1) + (x-1)(Bx+C)$

Equating

at $x = 1$ $\quad A = \frac{1}{2}$ $\quad$ the equating Coff. of x^2; $B = \frac{1}{2}$

$x = 0$ $\quad C = -\frac{1}{2}$ [1]

$$\frac{x^2-x+1}{(x-1)(x^2+1)} = \frac{1}{2}\frac{1}{x-1} + \frac{1}{2}\cdot\frac{x-1}{x^2+1}$$ [½]

$$I = \int\left[1 + \frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{x-1}{x^2+1}\right)\right]dx$$

$$= \int 1+\frac{1}{2}\left(\frac{1}{x-1}\right)+\frac{1}{2}\times\frac{1}{2}\left(\frac{2x}{x^2+1}\right)-\frac{1}{2}\left(\frac{1}{x^2+1}\right)dx$$ [½]

$$= x+\frac{1}{2}\log(x-1)+\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+C$$ [1]

$$= x+\frac{1}{4}\log(x-1)^2(x^2+1)-\frac{1}{2}\tan^{-1}x+C$$ [½]

35. Evaluate : $\int_0^{\pi/2}\frac{\cos^2 x dx}{1+3\sin^2 x}$ **[CBSE 2015, 4M]**

Sol. $I = \int_0^{\pi/2}\frac{\cos^2 x}{1+3\sin^2 x}dx$

$$I = \int_0^{\pi/2}\frac{1}{\sec^2 x+3\tan^2 x}$$

$$= \int_0^{\pi/2}\frac{1}{(1+4\tan^2 x)}\times\frac{\sec^2 x}{(1+\tan^2 x)}dx$$

$$= \int_0^{\infty}\frac{dt}{(1+t^2)(1+4t^2)}$$ [1]

Put $\tan x = t$

$\sec^2 x\,dx = dt$

$x = 0 \quad t = 0$

$x = \frac{\pi}{2} \quad t = \infty$

Put $t^2 = y$

$$\frac{1}{(1+t^2)(1+4t^2)} = \frac{1}{(1+y)(1+4y)} = \frac{A}{1+y}+\frac{B}{1+4y}$$

at $y = -1 \qquad A = -\frac{1}{3}$

$y = -\frac{1}{4} \qquad B = \frac{4}{3}$ [1]

$$\frac{1}{(1+y)(1+4y)} = -\frac{1}{3}\left(\frac{1}{1+y}\right)+\frac{4}{3}\left(\frac{1}{1+4y}\right)$$

Put $y = t^2$

$$\frac{1}{(1+t^2)(1+4t^2)} = -\frac{1}{3}\left(\frac{1}{1+t^2}\right)+\frac{4}{3}\left(\frac{1}{1+4t^2}\right)$$

$$I = \int_0^{\infty}\frac{1}{(1+t^2)(1+4t^2)}dt = -\frac{1}{3}\int_0^{\infty}\frac{1}{1+t^2}+\frac{4}{3}\int_0^{\infty}\frac{1}{1+(2t)^2}$$

$$= -\frac{1}{3}\left(\tan^{-1}t\right)_0^{\infty}+\frac{4}{3}\times\frac{1}{2}\left[\tan^{-1}2t\right]_0^{\infty}$$ [1]

$$= -\frac{1}{3}\left(\frac{\pi}{2}\right)+\frac{2}{3}\times\left(\frac{\pi}{2}\right)$$

$$= -\frac{\pi}{6}+\frac{\pi}{3} = \frac{\pi}{3}$$ [1]

36. Evaluate : $\int_0^{\pi/4}\left(\frac{\sin x+\cos x}{3+\sin 2x}\right)dx$ **[CBSE 2015, 4M]**

Sol. Let $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{3+\sin 2x}dx$

$$I=\int_0^{\pi/4}\frac{\sin x+\cos x}{4-(\sin x-\cos x)^2}$$

Put $\sin x-\cos x=t$

$(\cos x+\sin x)dx=dt$ **[1]**

$x=0 \quad t=-1$

$x=\frac{\pi}{4} \quad t=0$ **[1]**

$$I=\int_{-1}^{0}\frac{dt}{(2)^2-t^2}$$

$$I=\frac{1}{4}\left[\log\frac{2+t}{2-t}\right]_{-1}^{0}$$ **[1]**

$$I=\frac{1}{4}\left[\log 1-\log\frac{1}{3}\right]$$

$$I=-\frac{1}{4}\log\frac{1}{3}=\frac{1}{4}\log 3$$ **[1]**

37. Find : $\int(x+3)\sqrt{3-4x-x^2}\,dx$ **[CBSE 2016, 4M]**

Sol. $I=\int(x+3)\sqrt{3-4x-x^2}\,dx$

put $x+3=A\frac{d}{dx}(3-4x-x^2)+B$

$x+3=A(-4-2x)+B$

$A=-\frac{1}{2} \qquad B=1$ **[1]**

$\therefore \quad x+3=-\frac{1}{2}(-4-2x)+1$

$$I=-\frac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}\,dx+\int\sqrt{3-4x-x^2}\,dx$$ **[1]**

Let $I=I_1+I_2$ .... (i)

$$I_1=-\frac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}\,dx$$

Put $3-4x-x^2=t$

$(-4-2x)dx=dt$

$$I_1=-\frac{1}{2}\int\sqrt{t}.dt$$

$$I_1=-\frac{1}{2}.\frac{t^{3/2}}{3/2}+C_1=-\frac{1}{3}+t\sqrt{t}+C_1$$

$$I_1=-\frac{1}{3}(3-4x-x^2)\sqrt{3-4x-x^2}+C_1$$ .... (ii) **[1]**

$$I_2 = \int\sqrt{3-4x-x^2}\,dx$$

$$= \int\sqrt{-[x^2+4x-3]}\,dx$$

$$= \int\sqrt{-[(x+2)^2-(\sqrt{7})^2]}\,dx$$

$$= \int\sqrt{(\sqrt{7})^2-(x+2)^2}\,dx$$

$$= \frac{x+2}{2}\sqrt{(\sqrt{7})^2-(x+2)^2}+\frac{7}{2}\sin^{-1}\frac{(x+2)}{\sqrt{7}}+C_2$$

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$$I_2 = \frac{x+2}{2}\sqrt{3-4x-x^2}+\frac{7}{2}\sin^{-1}\frac{x+2}{\sqrt{7}}+C_2 \qquad \text{..........(iii)}$$

from equation (i), (ii) and (iii)

$$\therefore \quad I = -\frac{1}{3}(3-4x-x^2)\sqrt{3-4x+x^2}$$

$$+\frac{x+2}{2}\sqrt{3-4x-x^2}+\frac{7}{2}\sin^{-1}\frac{(x+2)}{\sqrt{7}}+C$$ [1]

where $C_1 + C_2 = C$

38. Evaluate : $\int_{-2}^{2}\frac{x^2}{1+5^x}dx$ **[CBSE 2016, 4M]**

Sol. Let $I = \int_{-2}^{2}\frac{x^2}{1+5^x}dx$ ....(1)

using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ [1]

$$I = \int_{-2}^{2}\frac{x^2}{1+5^{-x}}$$

$$I = \int_{-2}^{2}\frac{x^2 5^x}{1+5^x}dx \qquad \text{.........(2)}$$ [1]

$$\int_{-2}^{2}\left[\frac{x^2}{1+5^x}+\frac{x^2 5^x}{1+5^x}\right] = \int_{-2}^{2}\frac{x^2(1+5^x)}{1+5^x}$$

from equation (1) + (2)

$$2I = \int_{-2}^{2}x^2dx$$ [1]

$$2I = 2\int_0^2 x^2dx \qquad [\because\ x^2 \text{ is even function}]$$

$$I = \int_0^2 x^2dx = \frac{1}{3}(x^3)_0^2 = \frac{1}{3}$$

$$I = \frac{8}{3}$$ [1]

39. Find : $\int \frac{(2x-5)e^{2x}}{(2x-3)^3}dx$ **[CBSE 2016, 4M]**

OR

Find : $\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$

Sol. $I = \int e^{2x}\frac{(2x-5)}{(2x-3)^3}dx$

$$I = \int e^{2x}\left[\frac{(2x-3)}{(2x-3)^3} - \frac{2}{(2x-3)^3}\right]dx$$

$$I = \int e^{2x}\left[\frac{1}{(2x-3)^2} - \frac{2}{(2x-3)^3}\right]$$ **[1]**

Put $2x = t$

$$dx = \frac{dt}{2}$$

$$I = \frac{1}{2}\int e^t\left[\frac{1}{(t-3)^2} - \frac{2}{(t-3)^3}\right]dt$$ **[2]**

By prop. $\int e^x[f(x)+f'(x)]dx = e^x f(x) + C$

$$I = \frac{1}{2}\frac{e^t}{(t-3)^2} + C$$

$$I = \frac{1}{2}\frac{e^{2x}}{(2x-3)^2} + C$$ **[1]**

OR

$$I = \int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$$

Let $\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$ .... (1)

$x^2 + x + 1 = A(x^2+1) + (Bx+C)(x+2)$

at $x = -2$ $\quad A = \frac{3}{5}$

at $x = 0$ $\quad A + 2C = 1 \Rightarrow C = \frac{1}{5}$

equating the coeff. of x^2

$1 = A + B \Rightarrow \quad B = 1 - \frac{3}{5} = \frac{2}{5}$ **[1]**

Put in equation (1) and integrate

$$\int \frac{x^2+x+1}{(x+2)(x^2+1)}dx = \frac{3}{5}\int \frac{1}{x+2}dx + \int \frac{\frac{2}{5}x + \frac{1}{5}}{x^2+1}dx$$

$$= \frac{3}{5}\int \frac{1}{x+2}dx + \frac{2}{2.5}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$$ [2]

$$= \frac{3}{5}\log(x+2) + \frac{1}{5}\log(x^2+1) + \frac{1}{5}\tan^{-1}x + C$$ [1]

Long Answer : [6 Marks]

40. Evaluate $\int_0^3 (2x^2+3x+5)dx$ as limit of a sum. **[CBSE 2007, 6M]**

Sol. On comparing by $\int_a^b f(x)dx$

Here $a = 0, b = 3$, $f(x) = 2x^2 + 3x + 5$

$\because \int_a^b f(x)dx$

$= \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)]$ [1]

where $nh = b-a$

$\therefore\ nh = 3 - 0 = 3$ [1]

$f(a) = f(0) = 0+0+5 = 5$

$f(a+h) = f(0+h) = f(h) = 2h^2 + 3h + 5$

$f(a+2h) = f(0+2h) = f(2h) = 8h^2 + 6h + 5$

$f(a+(n-1)h) = f(0+(n-1)h) = f(n-1)h$

$= 2(n-1)^2h^2 + 3(n-1)h + 5$ [1]

$\therefore \int_0^3 (2x^2+3x+5)dx$

$$= \lim_{h\to 0} h\begin{bmatrix} 5+2h^2+3h+5+8h^2+6h+5 \\ +...+2(n-1)^2h^2+3(n-1)h+5 \end{bmatrix}$$

$$= \lim_{h\to 0} h\begin{bmatrix} 5n+2h^2(1+4+...+(n-1)^2) \\ +3h(1+2+...+(n-1)) \end{bmatrix}$$

$$= \lim_{h\to 0}\left[5n+2h^2\frac{n(n-1)(2n-1)}{6}+\frac{3hn(n-1)}{2}\right]$$ [1]

$$= \lim_{h\to 0} h\left[5nb+\frac{2nh(nh-h)(2nh-h)}{6}+\frac{3nh(nh-h)}{2}\right]$$

$$= \lim_{h\to 0}\left[5.3+\frac{2.3(3-h)(6-h)}{6}+\frac{3.3(3-h)}{2}\right] = \lim_{h\to 0}\left[15+(3-h)(6-h)+\frac{9(3-h)}{2}\right]$$ [1]

$$= 15 + 3.6 + \frac{9.3}{2} = 15 + 18 + \frac{27}{2} = \frac{93}{2}$$ [1]

41. Evaluate : $\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$ **[CBSE 2008, 6M]**

Sol. Let $I = \int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Put $x = a\tan^2 t \Rightarrow dx = 2a\tan t\sec^2 t\,dt$

$\therefore \quad I = \int_0^{\pi/4} \sin^{-1}\sqrt{\frac{a\tan^2 t}{a+a\tan^2 t}}\,.2a\tan t\sec^2 t\,dt$ **[1]**

$= a + a\tan^2 t$
$= a\sec^2 t$

$= \int_0^{\pi/4} \sin^{-1}\sqrt{\frac{a\tan^2 t}{a\sec^2 t}}\,.2a\tan t\sec^2 t\,dt$ **[1]**

$= \int_0^{\pi/4} \sin^{-1}\left(\frac{\sin t}{\cos t}.\cos t\right)2a\tan t\sec^2 t\,dt$

$= 2a\int_0^{\pi/4} \sin^{-1}\sin t.\tan t\sec^2 t\,dt$ **[1]**

tan t sec²x

sin²/cos t

$= 2a\int_0^{\pi/4} t\tan t\sec^2 t\,dt$

$= 2a\left[\left[t.\frac{\tan^2 t}{2}\right]_0^{\pi/4} - \int_0^{\pi/4} 1.\frac{\tan^2 t}{2}dt\right]$ **[1]**

$= 2a\left[\frac{\pi}{4}\frac{\tan^2.\frac{\pi}{4}}{2} - 0 - \frac{1}{2}\int_0^{\pi/4}\tan^2 t\,dt\right]$

$= 2a\left[\frac{\pi}{8} - \frac{1}{2}\int_0^{\pi/4}(\sec^2 t - 1)\,dt\right] = \frac{\pi a}{4} - a\left[\tan t - t\right]_0^{\pi/4}$ **[1]**

$= \frac{\pi a}{4} - a\left(\tan\frac{\pi}{4} - \frac{\pi}{4} - 0\right) = \frac{\pi a}{4} - a\left(1 - \frac{\pi}{4}\right)$

$= \frac{2\pi a}{4} - a = a\left(\frac{\pi-2}{2}\right)$ **[1]**

42. Evaluate : $\int_0^{\pi/2} \frac{x}{\sin x + \cos x}dx$ **[CBSE 2008, 6M]**

Sol. Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x}dx$ .....(1)

$I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$ **[1]**

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$$I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\cos x + \sin x}\,dx \quad .....(2)$$

On adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{x + \frac{\pi}{2} - x}{\sin x + \cos x}\,dx$$ [1]

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\frac{1}{\sqrt{2}}\,dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$$

$$2I = \frac{\pi}{2}\cdot\frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x}$$

$$2I = \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)}$$ [1]

$$2I = \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2} \operatorname{cosec}\left(x + \frac{\pi}{4}\right)dx$$

$$2I = \frac{\pi}{2\sqrt{2}}\left[\log\left|\operatorname{cosec}\left(x+\frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right)\right|\right]_0^{\frac{\pi}{2}}$$ [1]

$$2I = \frac{\pi}{2\sqrt{2}}\left[\begin{array}{l}\log\left|\operatorname{cosec}\left(\frac{\pi}{2}+\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{2}+\frac{\pi}{4}\right)\right| \\ \quad - \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right|\end{array}\right]$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right|\right]$$

$$\left[\because \operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \sec x \text{ and } \cot\left(\frac{\pi}{2}+x\right) = -\tan x\right]$$

$$2I = \frac{\pi}{2\sqrt{2}}\left[\log\left|\sqrt{2}+1\right| - \log\left|\sqrt{2}-1\right|\right]$$

$$2I = \frac{\pi}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$ [1]

$\int \operatorname{cosec} x$ Ans 2022

$\int \operatorname{cosec} x = \log|\operatorname{cosec} x - \cot x|$

$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$

$\cos\frac{\pi}{4}$ $\sin\frac{\pi}{4}$

$\sin(A+B)$

Intervals

$\int \operatorname{cosec} x = \log|\operatorname{cosec} x - \cot x|$

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$$2I = \frac{\pi}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}\right|$$

$$2I = \frac{\pi}{2\sqrt{2}} \log\left|\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right| = \frac{\pi}{2\sqrt{2}} .2\log\left|\sqrt{2}+1\right|$$

$$\therefore \quad I = \frac{\pi}{2\sqrt{2}} \log\left(\sqrt{2}+1\right)$$ [1]

43. Evaluate : $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2 \sin^2 x}$ **[CBSE 2009, 6M]**

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Sol. $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$

$I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2\cos^2 x + b^2\sin^2 x}$ [By property] $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ [1]

$2I = \int_0^{\pi} \frac{\pi dx}{a^2\cos^2 x + b^2\sin^2 x}$ [1]

$I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$I = \frac{\pi}{2} \times 2\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$ [By property] $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a - x) = f(x)$ [1]

dividing by $\cos^2 x$

$I = \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$ [1]

Let $\tan x = t$

$\Rightarrow \quad \sec^2 x\,dx = dt$

$$I = \pi\int_0^{\infty} \frac{dt}{a^2 + b^2t^2} = \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{t^2 + \left(\frac{a}{b}\right)^2}$$ [1]

$$I = \left(\frac{\pi}{b^2}\right)\frac{1}{\left(\frac{a}{b}\right)}\left|\tan^{-1}\left(\frac{t}{a/b}\right)\right|_0^{\infty} = \frac{\pi}{ab}.\frac{\pi}{2} = \frac{\pi^2}{2ab}.$$ [1]

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44. Evaluate : $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$ **[CBSE 2011, 6M]**

Sol. $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx = \int \frac{6x+7}{\sqrt{x^2 - 9x + 20}}$

Let $6x + 7 = \lambda \frac{d}{dx}(x^2 - 9x + 20) + \mu$

$6x + 7 = 2\lambda x - 9\lambda + \mu$

$\Rightarrow \lambda = 3 \qquad 7 = \mu - 9\lambda \Rightarrow \mu = 34$ [2]

$$I = \int \frac{3(2x-9)}{\sqrt{x^2 - 9x + 20}} dx + 34 \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}}$$ [1]

$$= \int \frac{3}{\sqrt{t}} dt + 34 \log\left|x - \frac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + c$$ [1]

$$= 6\sqrt{t} + 34.\log\left|x - \frac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + c$$

$$= 6\sqrt{x^2 - 9x + 20} + 34 \log\left|x - \frac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + c$$ [2]

45. Evaluate : $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ **[CBSE 2012, 6M]**

Sol. Let $I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Put $\sin^{-1}x = t \Rightarrow x = \sin t$

$\Rightarrow dx = \cos t\, dt$ [1]

$$I = \int \frac{t \sin t}{\sqrt{1 - \sin^2 t}} \times \cos t\, dt$$

$$I = \int t \sin t\, dt$$ [1]

Integrating by parts

$$I = t \times \int \sin t\, dt - \int\left(\frac{dt}{dt} \times \int \sin t\, dt\right) dt$$ [1]

$$= t \times -\cos t - \int 1 \times -\cos t\, dt = -t\cos t + \int \cos t\, dt = -t\cos t + \sin t + C$$ [1]

$$I = -\sqrt{1-x^2} \sin^{-1}x + x + C$$

$$I = x - \sqrt{1-x^2} \sin^{-1}x + C$$ [2]

46. Evaluate : $\int \frac{x^2+1}{(x-1)^2(x+3)} dx$ **[CBSE 2012, 6M]**

Sol. We have, $I = \int \frac{x^2+1}{(x-1)^2(x+3)} dx$

Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$ ...(i) [1]

$\Rightarrow \quad x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$ ...(ii)

Putting $x - 1 = 0$ i.e. $x = 1$ in (ii), we get

$$2 = 4B \quad \Rightarrow \quad B = \frac{1}{2}$$ [1]

Putting $x + 3 = 0$ i.e. $x = -3$ in (ii), we get

$$10 = 16C \Rightarrow \quad C = \frac{5}{8}$$

Equating the coefficients of x^2 on both sides of the identity (ii), we get

$$1 = A + C \Rightarrow \quad A = 1 - C = 1 - \frac{5}{8} = \frac{3}{8}$$ [1]

Substituting the values of A, B and C in (i), we get

$$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{5}{8}(x+3)$$

$$\Rightarrow \quad I = \int \frac{x^2+1}{(x-1)^2(x+3)}dx = \frac{3}{8}\int\frac{1}{x-1}dx + \frac{1}{2}\int\frac{1}{(x-1)^2}dx + \frac{5}{8}\int\frac{1}{x+3}dx$$ [2]

$$\Rightarrow \quad I = \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + C$$ [1]

47. Evaluate : $\int \frac{1}{\cos^4 x + \sin^4 x}dx$ **[CBSE 2014, 6M]**

Sol. $\int \frac{dx}{\cos^4 x + \sin^4 x}$

$$= \int \frac{\frac{1}{\cos^4 x}dx}{1 + \tan^4 x}$$

$$= \int \frac{\sec^2 x \sec^2 x dx}{1 + \tan^4 x}$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x dx}{1+\tan^4 x}$$

put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$ [2]

$$= \int \frac{(1+t^2)dt}{1+t^4}$$

$$= \int \frac{\left(\frac{1}{t^2}+1\right)dt}{\frac{1}{t^2}+t^2} \quad \{\text{dividing each by } t^2\}$$

$\left(\frac{1}{t^2}+t^2\right)$

$\left(t-\frac{1}{t}\right)^2 = t^2 + \frac{1}{t^2} - 2t\times\frac{1}{t}$

$$= \int \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+2}$$

put $t-\frac{1}{t}=z \quad \Rightarrow \quad \left(1+\frac{1}{t^2}\right)dt = dz$ **[2]**

$$= \int \frac{dz}{z^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\left(t-\frac{1}{t}\right)+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\left(\tan x - \frac{1}{\tan x}\right)+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{2}}\right)+C$$ **[2]**

48. Evaluate : $\int(\sqrt{\cot x}+\sqrt{\tan x})dx$ **[CBSE 2014, 6M]**

Sol. Let $I = \int(\sqrt{\cot x}+\sqrt{\tan x})dx$ $= \int \frac{\sqrt{\cos x}}{\sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x}}$ $\quad \int \frac{\cos x + \sin x}{\sqrt{\sin x \cos x}}$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}dx$$

$\sqrt{\sin 2x}$ $\quad \sin x \quad - \cos$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}dx$$ **[2]**

Put $\sin x - \cos x = t$

$(\cos x + \sin x)dx = dt$

$$= \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$ **[2]**

$$= \sqrt{2}[\sin^{-1} t]$$

$$= \sqrt{2}[\sin^{-1}(\sin x - \cos x)] + C$$ **[2]**

Aliter :

We have $I = \int\left[\sqrt{\cot x}+\sqrt{\tan x}\right]dx = \int\sqrt{\tan x}(1+\cot x)dx$

Put $\tan x = t^2$, so that $\sec^2 x dx = 2t dt$

or $\quad dx = \frac{2t\,dt}{1+t^4}$ **[1]**

Then $\quad I = \int t\left(1+\frac{1}{t^2}\right)\frac{2t}{(1+t^4)}dt$

$$= 2\int\frac{(t^2+1)}{t^4+1}dt = 2\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)} = 2\int\frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+2}$$ **[2]**

Put $t-\frac{1}{t}=y$, so that $\left(1+\frac{1}{t^2}\right)dt = dy$. Then [1]

$$I = 2\int \frac{dy}{y^2+\left(\sqrt{2}\right)^2} = \sqrt{2}\tan^{-1}\frac{y}{\sqrt{2}} + C = \sqrt{2}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}} + C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + C = \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\tan x}\right) + C$$ [2]

49. Evaluate : **[CBSE 2014, 6M]**

$$\int \frac{1}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x}\,dx$$

Sol. $I = \int \frac{1}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x}\,dx$

$I = \int \frac{\sec^2 x(1+\tan^2 x)}{\tan^4 x + \tan^2 x + 1}$ (Put $\tan x = t$, $\sec^2 x\,dx = dt$) [1]

$I = \int \frac{(1+t^2)}{t^4+t^2+1}\,dt$

$I = \int \frac{(1+1/t^2)}{t^2+\frac{1}{t^2}+1}\,dt$

$I = \int \frac{(1+1/t^2)}{(t-1/t)^2+(\sqrt{3})^2}\,dt$ $\left(\text{Put } t-\frac{1}{t}=u\ \left(1+\frac{1}{t^2}\right)dt = du\right)$ [2½]

$I = \int \frac{du}{u^2+(\sqrt{3})^2}$

$I = \frac{1}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + C$

$I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{3}\tan x}\right) + C$ [2½]

NCERT IMPORTANT QUESTIONS	
Examples	4, 6(ii), 10(ii), 12, 15, 21, 22(ii), 26, 30, 32, 36, 39, 40, 43
Exercise # 7.2	24, 36, 37, 38
Exercise # 7.3	14, 19, 22
Exercise # 7.4	7, 15, 19
Exercise # 7.5	6, 9, 12 16, 21
Exercise # 7.6	6, 10, 18, 20, 22
Exercise # 7.7	5, 9
Exercise # 7.8	6
Exercise # 7.9	18
Exercise # 7.10	8, 9, 10
Exercise # 7.11	8, 9, 12, 16, 19
Miscellaneous Exercise	5, 10, 18, 19, 21, 24, 31, 33, 40, 43, 44