MISCELLANEOUS EXAMPLES

Very Short Answer : [1 Mark]

1. Evaluate : $\begin{vmatrix} \sin 30^\circ & \cos 30^\circ \\ -\sin 60^\circ & \cos 60^\circ \end{vmatrix}$ **[CBSE 2008, 1M]**

Sol. Let $\Delta = \begin{vmatrix} \sin 30^\circ & \cos 30^\circ \\ -\sin 60^\circ & \cos 60^\circ \end{vmatrix}$

$= \sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ$

$= \sin(30^\circ + 60^\circ) = \sin 90^\circ = 1$ **[1]**

2. Write the value of the determinant : $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ **[CBSE 2009, 1M]**

Sol. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$\begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0$ **[1]**

3. Find the value of x from the following : $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$ **[CBSE 2009, 1M]**

Sol. $2x^2 - 8 = 0 \quad \Rightarrow \quad 2x^2 = 8$

$\Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$ **[1]**

4. What positive value of x makes the following pair of determinants equal ?

$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}, \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$ **[CBSE 2010, 1M]**

Sol. $\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix} = \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix} \Rightarrow 2x^2 - 15 = 32 - 15$

$\Rightarrow \quad x^2 = 16 \quad \Rightarrow \quad x = \pm 4$

$\Rightarrow \quad x = 4 \quad \because \quad x = -4$ (Rejected) **[1]**

5. Evaluate : $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$ **[CBSE 2011, 1M]**

Sol. $\Delta = \cos 15^\circ \cos 75^\circ - \sin 15^\circ \sin 75^\circ$

$= \cos(15^\circ + 75^\circ) = \cos 90^\circ = 0$ **[1]**

6. Let A be a square matrix of order 3×3. Write the value of $|2A|$, where $|A| = 4$. **[CBSE 2012 1M]**

Sol. Given A be a square matrix of order 3×3 and $|A| = 4$

$\therefore |2A| = 2^3|A| = 8 \times 4 = 32$ **[1]**

7. If A_{ij} is the cofactor of the elements a_{ij} of the detereminant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32}.A_{32}$. **[CBSE 2013 1M]**

Sol. $a_{32} = 5,\ A_{32} = -\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = -(8-30) = 22$

$\therefore a_{32}.A_{32} = 5 \times 22 = 110$ **[1]**

8. For what values of k, the system of linear equations

$x + y + z = 2$

$2x + y - z = 3$

$3x + 2y + kz = 4$

has a unique solution ? **[CBSE 2016, 1M]**

Sol. Matrix from of system of linear equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{bmatrix}$$

System of linear equation's has unique solution.

If $|A| \neq 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$ **[½]**

$\Rightarrow \quad 1(k + 2) - 1(2k + 3) + 1(4 - 3) \neq 0$

$\Rightarrow \quad k \neq 0$ **[½]**

9. If A is a 3×3 matrix and $|3A| = k|A|$, then write the value of k. **[CBSE 2016, 1M]**

Sol. $|3A| = 3^3|A|$

$k = 27$ **[1]**

Short Answer : [4 Marks]

10. If a, b, c are in A.P., show that : $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$ **[CBSE 2006, 4M]**

Sol. Since a, b and c are in A.P., we have

Now, $2b = a + c$ .....(1)

a b c in AP

2b = a + c

$$\text{L.H.S.} = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$ **[1]**

operate : $R_1 \to R_1 + R_3 - 2R_2$

$$= \begin{vmatrix} 0 & 0 & a+c-2b \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$ **[1½]**

$$= \begin{vmatrix} 0 & 0 & 0 \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0 = \text{R.H.S.}$$ **[1½]**

11. Using the properties of determinants, prove that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

[CBSE 2007, Set-I, 4M]

Sol. Here, $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 - C_3$

$$= \begin{vmatrix} 2b & b+c & c+a \\ 2c & c+a & a+b \\ 2a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} b & b+c & c+a \\ c & c+a & a+b \\ a & a+b & b+c \end{vmatrix}$$ **[2]**

$(C_2 \to C_2 - C_1)$

$$= 2\begin{vmatrix} b & c & c+a \\ c & a & a+b \\ a & b & b+c \end{vmatrix}$$ **[1]**

$(C_3 \to C_3 - C_2)$

$$= 2\begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \quad (R_1 \leftrightarrow R_3 \text{ then } R_2 \leftrightarrow R_3)$$ **[1]**

12. Using the properties of determinants, prove that $\begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0$, where α, β and γ are in A.P.

[CBSE 2007,Set-II, 4M]

Sol. α, β, and γ are in AP.

$\therefore 2\beta = \alpha + \gamma$

$2\beta = \alpha + \gamma$

$$\text{LHS} = \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 3x-6 & 3x-9 & 3x-\alpha-\beta-\gamma \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix}$$ [1]

$$= \begin{vmatrix} 3(x-2) & 3(x-3) & 3x-\beta-2\beta \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix}$$ [1]

$$= \begin{vmatrix} 3(x-2) & 3(x-3) & 3(x-\beta) \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 3\begin{vmatrix} x-2 & x-3 & x-\beta \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix}$$ [2]

$= 3 \times 0 = 0$

[$\because$ The elements of R_1 and R_2 are same]

13. Using the properties of determinants, prove that $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$.

[CBSE 2007,Set-III, 4M]

Sol. $$\text{LHS} = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix} = (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$$ [2]

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & 4-x & 0 \\ 0 & 0 & 4-x \end{vmatrix} = (5x+4)(4-x)^2$$ [2]

14. If x, y, z are different and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, show that $xyz = -1$. **[CBSE 2008, 4M]**

Sol. We have

a B C

2B = a + c

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$ [1]

$$= (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} (1 + xyz)$$ [1]

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} (1 + xyz)$$ [1]

$$= (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} (1 + xyz)$$

$= (y - x)(z - x)[1(z + x - y - x)](1 + xyz)$

$= (y - x)(z - x)(z - y)(1 + xyz)$

$= (1 + xyz)(x - y)(y - z)(z - x)$

Since $\Delta = 0$ and x, y, z are all different

i.e. $x - y \neq 0$, $y - z \neq 0$ and $z - x \neq 0$

Hence, we get $1 + xyz = 0$ [1]

15. Using properties of determinants, prove the following : **[CBSE 2008, 4M]**

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

Sol. LHS $= \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$

Applying $R_1 \to \frac{1}{a}R_1, R_2 \to \frac{1}{b}R_2$ and $R_3 \to \frac{1}{c}R_3$

$$= abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$ [1]

Applying $C_1 \to aC_1, C_2 \to bC_2$ and $C_3 \to cC_3$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2+1 \\ a^2 & b^2 & c^2 \end{vmatrix}$$ [1]

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2+1 \\ 1+a^2+b^2+c^2 & b^2 & c^2 \end{vmatrix}$$ [1]

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$= (1+a^2+b^2+c^2)\,[1(1-0)]$$

$$= 1+a^2+b^2+c^2 = \text{R.H.S.}$$ [1]

16. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$$

[CBSE 2009, 4M]

Sol. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1+p & 1+p \\ 2 & 3+2p & 1+3p \\ 3 & 6+3p & 1+6p \end{vmatrix} + q \begin{vmatrix} 1 & 1+p & 1 \\ 2 & 3+2p & 2 \\ 3 & 6+3p & 3 \end{vmatrix} \quad \left[\because \begin{matrix} C_1 \,\&\, C_3 \\ \text{are same} \end{matrix}\right]$$ [1]

$$= \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 1+3p \\ 3 & 6 & 1+6p \end{vmatrix} + p \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 2 & 1+3p \\ 3 & 3 & 1+6p \end{vmatrix} \quad \left[\because \begin{matrix} C_1 \,\&\, C_2 \\ \text{are same} \end{matrix}\right]$$ [1]

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 6 & 1 \end{vmatrix} + p \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 6 & 6 \end{vmatrix} \quad \left[\because \begin{matrix} C_2 \,\&\, C_3 \\ \text{are same} \end{matrix}\right]$$ [1]

$(C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 3 & -2 \end{vmatrix} \qquad \text{Expand along } R_1$$

$= 1.(-2+3) = 1$ [1]

17. Using properties of determinants, solve for x : $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$. **[CBSE 2005, 2011 4M]**

Sol. $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

operate : $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0$$ [1]

$$\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0$$

operate : $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$$ [1]

$$\Rightarrow (3a-x)\,.\,1\,.\begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix} = 0$$ [1]

$\Rightarrow (3a-x)\,.\,2x\,.\,2x = 0$

$\Rightarrow x^2(3a-x) = 0 \quad \Rightarrow \quad x = 0, 3a.$ [1]

18. Using properties of determinants, show that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$ **[CBSE 2012 4M]**

Sol. Let $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \to R_1 - (R_2 + R_3)$ to Δ, we get

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 2\begin{vmatrix} 0 & -c & -b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$ **[1]**

Applying $R_2 \to R_2 + R_1$ & $R_3 \to R_3 + R_1$

$$= 2\begin{vmatrix} 0 & -c & -b \\ b & a & 0 \\ c & 0 & a \end{vmatrix}$$ **[2]**

expanding w.r.t. R_1

$2[0 + c(ab - 0) - b(0 - ac) = 4abc.$ **[1]**

19. Using properties of determinants, prove the following :

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$$ **[CBSE 2013 4M]**

Sol. $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 3(x+y) & x+y & x+2y \\ 3(x+y) & x & x+y \\ 3(x+y) & x+2y & x \end{vmatrix}$$ **[1½]**

$$= 3(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$$

$R_2 \to R_2 - R_1 / R_3 \to R_3 - R_1$

$$= 3(x+y)\begin{vmatrix} 1 & x+3 & x+2y \\ 0 & -y & -y \\ 0 & y & -2y \end{vmatrix}$$

$$= 3y^2(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 0 & -1 & -1 \\ 0 & 1 & -2 \end{vmatrix}$$ [1½]

$= 3y^2(x + y)1 \times [+2 + 1]$

$= 9y^2(x + y) = \text{RHS}.$ [1]

20. Using properties of determinants, prove the following :

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+zx)$$ **[CBSE 2006, 2013, 4M]**

Sol. $$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$ [1½]

$$\Rightarrow (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\Rightarrow (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & -y+x \\ 0 & -z+x & 2z+x \end{vmatrix}$$ [1½]

$\Rightarrow (x + y + z)1\times[(2y + x)(2z + x) - (x - y)(x - z)]$

$\Rightarrow (x + y + z) [(4yz + 2xy + 2zx + x^2) - (x^2 - xz - xy + yz)]$

$\Rightarrow (x + y + z)[3xy + 3yz + 3zx] \quad \Rightarrow \quad 3(x + y + z)(xy + yz + zx)$ [1]

21. Using properties of determinants, prove that : $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab$

[CBSE 2008, 2014 4M]

Sol. L.H.S. $= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$C_1 \to \frac{1}{a}\times C_1,\ C_2 \to \frac{1}{b}\times C_2,\ C_3 \to \frac{1}{c}\times C_3$

$$= abc\begin{vmatrix} \frac{1+a}{a} & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1+b}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & \frac{1+c}{c} \end{vmatrix}$$ [1]

(Operate : $C_1 \to C_1 + C_2 + C_3$)

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$ [1]

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 1 & 1+\frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$

(Operate : $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$).

$$= (abc + bc + ca + ab)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ [1]

$$= (abc + bc + ca + ab) . 1 . \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$= (abc + bc + ca + ab)$. [1]

22. Using properties of determinants, prove that

$$\begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix} = 2(a-b)(b-c)(c-a)(a+b+c).$$ **[CBSE 2015, 4M]**

Sol. $\begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix}$

$$= 2\begin{vmatrix} a^3 & 1 & a \\ b^3 & 1 & b \\ c^3 & 1 & c \end{vmatrix}$$ [½]

applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$=2\begin{vmatrix} a^3 & 1 & a \\ b^3-a^3 & 0 & b-a \\ c^3-a^3 & 0 & c-a \end{vmatrix}$$ [1]

$$=2(b-a)(c-a)\begin{vmatrix} a^3 & 1 & a \\ b^2+ba+a^2 & 0 & 1 \\ c^2+ca+a^2 & 0 & 1 \end{vmatrix}$$ [½]

$\Rightarrow \quad R_3 \to R_3 - R_2$

$$\Rightarrow \quad 2(b-a)(c-a)\begin{vmatrix} a^3 & 1 & a \\ b^2+ba+a^2 & 0 & 1 \\ (c-b)(b+c)+a(c-b) & 0 & 0 \end{vmatrix}$$ [1]

$$\Rightarrow \quad 2(b-a)(c-a)(c-b)\begin{vmatrix} a^3 & 1 & a \\ b^2+ba+a^2 & 0 & 1 \\ a+b+c & 0 & 0 \end{vmatrix}$$

$= 2[(b-a)(c-a)(c-b)] \times -1[0-(a+b+c)]$

$= 2(a-b)(b-c)(c-a)(a+b+c)$ [1]

Long Answer : [6 Marks]

23. Using properties of determinants, prove the following : **[CBSE 2010, 6M]**

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)\,(x-y)\,(y-z)\,(z-x)$$

Sol. $$\text{L.H.S.}=\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}=\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+\begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$ [1]

$$=\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$ [1]

$$=(1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$ [1]

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$

$$=(1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$

$$=(1+pxyz)(y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$ [2]

$=(1+pxyz)(x-y)(z-x)(y-z)$ [1]

24. If a, b, c are positive and unequal, show that the following determinant is negative :

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

[CBSE 2010, 6M]

Sol. $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$$ **[1]**

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$ **[1]**

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_2$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-c \\ c & a-c & b-a \end{vmatrix}$$ **[2]**

$= (a + b + c)\ [(c - b)(b - a) - (a - c)^2]$

$= (a + b + c)[bc - ac - b^2 + ab - a^2 - c^2 + 2ac]$

$= (a + b + c)[ab + bc + ac - a^2 - b^2 - c^2]$

$= -\ (a + b + c)\ [a^2 + b^2 + c^2 - ab - bc - ac]$ **[1]**

$= -\dfrac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

$= -\ (a^3 + b^3 + c^3 - 3abc)$ **[1]**

25. Using properties of determinants, prove that $\begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (z+y)^2 & xy \\ zy & xy & (z+x)^2 \end{vmatrix} = 2xyz(x+y+z)^3$.

[CBSE 2016, 6M]

OR

If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ and $A^3 - 6A^2 + 7A + kI_3 = O$ find k.

Sol. $\text{LHS} = \begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (z+y)^2 & xy \\ zy & xy & (z+x)^2 \end{vmatrix}$

Applying $R_1 \to zR_1$, $R_2 \to xR_2$, $R_3 \to yR_3$ and dividing by xyz.

$$\Rightarrow \frac{1}{xyz}\begin{vmatrix} z(x+y)^2 & z^2x & z^2y \\ zx^2 & x(z+y)^2 & x^2y \\ zy^2 & xy^2 & y(z+x)^2 \end{vmatrix}$$ [1]

Taking common factors z, x, y from C_1, C_2, C_3 respectively.

$$\Rightarrow \frac{xyz}{xyz}\begin{vmatrix} (x+y)^2 & z^2 & z^2 \\ x^2 & (z+y)^2 & x^2 \\ y^2 & y^2 & (z+x)^2 \end{vmatrix}$$ [1]

$\Rightarrow$ apply $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$

$$\Rightarrow \begin{vmatrix} (x+y)^2 - z^2 & 0 & z^2 \\ 0 & (z+y)^2 - x^2 & x^2 \\ y^2 - (z+x)^2 & y^2 - (z+x)^2 & (z+x)^2 \end{vmatrix}$$

Taking common factor $(x + y + z)$ from C_1 & C_2

$$\Rightarrow (x+y+z)^2\begin{vmatrix} x+y-z & 0 & z^2 \\ 0 & z+y-x & x^2 \\ y-(z-x) & y-z-x & (z+x)^2 \end{vmatrix}$$ [1]

$R_3 \to R_3 - (R_1 + R_2)$

$$\Rightarrow (x+y+z)^2\begin{vmatrix} x+y-z & 0 & z^2 \\ 0 & z+y-x & x^2 \\ -2x & -2z & 2zx \end{vmatrix}$$ [1]

$C_1 \to C_1 + \frac{1}{z}C_3$ $\qquad C_2 \to C_2 + \frac{1}{x}C_3$

$$\Rightarrow (x+y+z)^2\begin{vmatrix} x+y & z^2/x & z^2 \\ x^2/z & z+y & x^2 \\ 0 & 0 & 2zx \end{vmatrix}$$ [1]

Expanding along R_3

$\Rightarrow$ $(x + y + z)^2.2zx[(x + y)(z + y) - xz]$ [1]

$\Rightarrow$ $2xyz(x + y + z)^3$

OR

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$ [1½]

$$A^3 = A^2.A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

Now $A^3 - 6A^2 + 7A + KI = 0$

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2+k & 0 & 0 \\ 0 & -2+k & 0 \\ 0 & 0 & -2+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad -2 + k = 0$

$k = 2$

Done

NCERT IMPORTANT QUESTIONS	
Examples	5, 15, 16, 32
Exercise # 4.2	10, 11, 13
Exercise # 4.3	2, 3
Exercise # 4.4	5
Exercise # 4.5	12, 13, 15, 18
Exercise # 4.5	15
Miscellaneous Exercise	5, 6, 11, 12, 16, 19