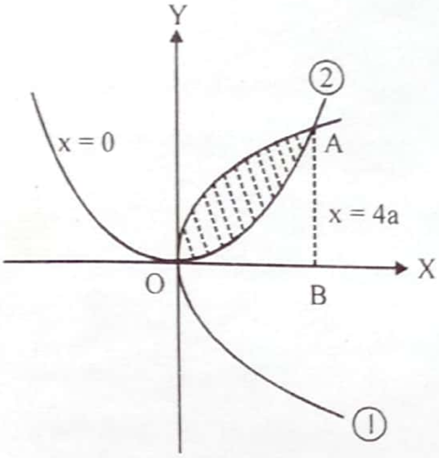
MISCELLANEOUS EXAMPLES

Long Answer : [6 Marks]

1. Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. **[CBSE 2004, 2008, 6M]**

Sol. $y^2 = 4ax$ ...(1)

$x^2 = 4ay$ ...(2)



For their points of intersection,

$$x = \frac{y^2}{4a} \quad (a > 0)$$

$$\Rightarrow \left(\frac{y^2}{4a}\right)^2 = 4ay \Rightarrow y^4 - 64a^3y = 0$$

$$\Rightarrow y[y^3 - (4a)^3] = 0 \quad \Rightarrow \quad y = 0, 4a$$

When $y = 0$, $x = 0$

When $y = 4a$, $x = 4a$

$\therefore$ the two points of intersection of (1) and (2) are O(0, 0) and A(4a, 4a). [1]

The area of the region included between (1) and (2) = Area of the shaded region

$$= \int_0^{4a} (y_1 - y_2)dx = \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] dx$$ [1]

$$= \left[2\sqrt{a}.\frac{x^{3/2}}{3/2} - \frac{1}{4a}.\frac{x^3}{3}\right]_0^{4a}$$ [1]

$$= \frac{4}{3}\sqrt{a}(4a)^{3/2} - \frac{1}{12a}.(4a)^3 - 0$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \text{ sq. units.}$$ [2]

Sol. $x^2 + y^2 = 16$ ...(i)

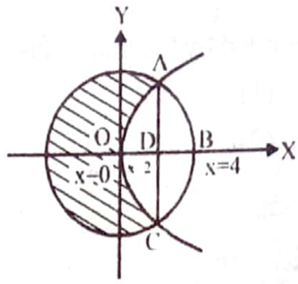
$\therefore \quad y = \sqrt{16 - x^2}$

centre = (0,0)

radius = 4

$y^2 = 6x$

$\therefore \quad y = \sqrt{6x}$



[1]

Solving equation (i) and (ii), we get

$x^2 + 6x = 16$

$x^2 + 6x - 16 = 0$

$x = 2, x = -8$ [1]

Area of region bounded by OAC

$$A_1 = 2\int_0^2 \sqrt{6x}\,dx = 2\sqrt{6}\int_0^2 x^{\frac{1}{2}}dx = 2\sqrt{6}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2$$

$$= 2\sqrt{6}.\frac{2}{3}\left[2^{\frac{3}{2}} - 0\right] = \frac{4}{3}\sqrt{6}.2\sqrt{2} = \frac{8}{3}\sqrt{12}$$

$$= \frac{8}{3}2\sqrt{3} = \frac{16}{3}\sqrt{3} \text{ sq. unit}$$ [1½]

Area of region bounded by ABC

$$A_2 = 2\int_2^4 \sqrt{16 - x^2}\,dx = 2\int_2^4 \sqrt{(4)^2 - x^2}\,dx$$

$$A_2 = 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{(4)^2}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= 2\left[0 + \frac{16}{2}\sin^{-1}\frac{4}{4} - \frac{2}{2}\sqrt{16 - 4} - \frac{16}{2}\sin^{-1}\frac{2}{4}\right]$$

$$= 2\left[8\sin^{-1}1 - \sqrt{12} - 8\sin^{-1}\frac{1}{2}\right]$$

$$= 2\left[8.\frac{\pi}{2} - 2\sqrt{3} - 8.\frac{\pi}{6}\right] = 2\left[4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right]$$

$$= 2\left[\frac{8\pi}{3} - 2\sqrt{3}\right] = \frac{16\pi}{3} - 4\sqrt{3} \text{ sq. unit}$$ [1½]

Area of region bounded by OABC = $A_1 + A_2$

$$A = \frac{16}{3}\sqrt{3} + \frac{16\pi}{3} - 4\sqrt{3} = \frac{16\pi}{3} + \frac{4}{3}\sqrt{3} \text{ sq.unit}$$

Exterior area to the parabola = area of circle – A

$$= \pi(4)^2 - \left(\frac{16\pi}{3} + \frac{4}{3}\sqrt{3}\right) = 16\pi - \frac{16\pi}{3} - \frac{4}{3}\sqrt{3}$$

$$= \left(\frac{32\pi}{3} - \frac{4}{3}\sqrt{3}\right) \text{ sq. unit}$$ [1]

3. Using integration, find the area lying above the x-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the the parabola $y^2 = 4x$. **[CBSE 2008, 6M]**

Sol. The given equation of circle

$x^2 + y^2 = 8x$ can be written as

$x^2 + y^2 - 8x = 0$

$(x - 4)^2 + y^2 - 16 = 0$

$(x - 4)^2 + y^2 = 16$ ... (i)

Therefore, the centre of circle is C(4, 0) and radius 4.

The given equation of parabola is

$y^2 = 4x$ ... (ii)

Solving (1) and (2), we get

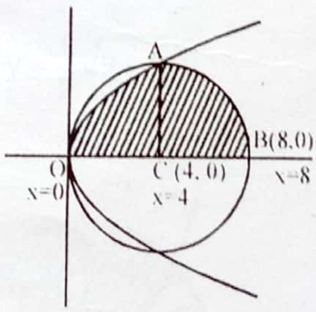
$(x - 4)^2 + 4x = 16$

$x^2 + 16 - 8x + 4x = 16$

$x^2 - 4x = 0$

$x(x - 4) = 0$

$\therefore$ x = 0, x = 4 [1]



[1]

The required area of the region OABCO included between these two curves above x-axis is

= (area of the region OACO) + (Area of the region CABC)

$$= \int_0^4 y(\text{parabola})\,dx + \int_4^8 y(\text{circle})\,dx$$ [1]

$$= \int_0^4 2\sqrt{x}\,dx + \int_4^8 \sqrt{(16 - (x-4)^2}\,dx$$ [1]

$$= 2\int_0^4 x^{1/2}\,dx + \int_4^8 \sqrt{(4)^2 - (x-4)^2}\,dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^4 + \left[(x-4)\sqrt{(4)^2-(x-4)^2} + \frac{(4)^2}{2}\sin^{-1}\frac{x-4}{4}\right]_4^8$$ [1]

$$= \frac{4}{3}[(2^2)^{3/2}] + \left[0 + 8\sin^{-1}\left(\frac{8-4}{4}\right) - 0\right]$$

$$= \frac{4}{3}.8 + 8\sin^{-1}1 = \frac{32}{3} + 8.\frac{\pi}{2}$$

$$= \frac{32}{3} + 4\pi = \frac{4}{3}(8+3\pi) \text{ sq. unit.}$$ [1]

4. Using integration, find the area of the triangular region whose vertices are (1, 0), (2, 2) and (3, 1). **[CBSE 2008, 6M]**

Sol. Let A(1, 0), B(2, 2) and C(3, 1) be the vertices of ΔABC

Equation of line AB is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \Rightarrow y - 0 = \frac{2-0}{2-1}(x-1)$$

$$\Rightarrow y = 2(x-1)$$

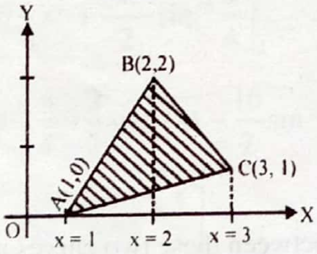
Equation of BC is $y - 2 = \frac{1-2}{3-2}(x-2)$

$$\Rightarrow y - 2 = -x + 2 \Rightarrow y = 4 - x$$ [1]

and equation of CA is $y - 1 = \frac{1-0}{3-1}(x-3)$

$$\Rightarrow y - 1 = \frac{1}{2}(x-3) \Rightarrow y = \frac{1}{2}x - \frac{3}{2} + 1$$ [1]

$$\Rightarrow y = \frac{1}{2}x - \frac{1}{2} \Rightarrow y = \frac{1}{2}(x-1)$$ [1]



[1]

Hence area of ΔABC

$$= \int_1^2 y(\text{line AB})dx + \int_2^3 y(\text{line BC})dx - \int_1^3 y(\text{line CA})dx$$ [1]

$$= \int_1^2 2(x-1)dx + \int_2^3 (4-x)dx - \int_1^3 \frac{1}{2}(x-1)dx$$

$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$=2\left[0+\frac{1}{2}\right]+\left[\frac{15}{2}-6\right]-\frac{1}{2}\left[\frac{3}{2}+\frac{1}{2}\right]$$

$$=1+\frac{3}{2}-1=\frac{3}{2}\text{ sq. units}$$ [1]

5. Using integration, find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$

[CBSE 2009, 6M]

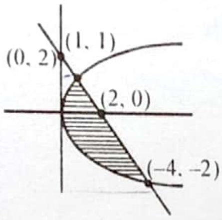
Sol. $y^2 = 2-y$

$y^2 + y-2 = 0$

$(y + 2)(y-1) = 0$

$y = -2, 1$ [1½]

$= x = 4, 1$



$$\text{Area} = \int_{-2}^{1}(2-y-y^2)dy$$ [1½]

$$=\left[2y-\frac{y^2}{2}-\frac{y^3}{3}\right]_{-2}^{1}$$ [1½]

$$=2-\frac{1}{2}-\frac{1}{3}-\left(-4-2+\frac{8}{3}\right)=5-\frac{1}{2}=\frac{9}{2}\text{ sq. units}$$ [1½]

6. Using integration, find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

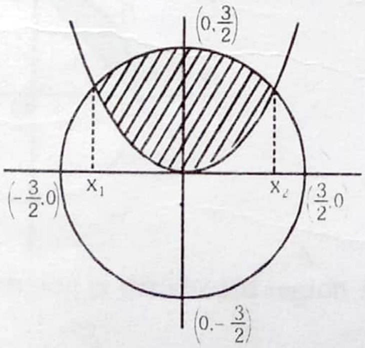
[CBSE 2010, 6M]

OR

Using integration, find the area of the triangle ABC, coordinates of whose vertices are A(4,1), B(6,6) and C(8,4).

Sol. $4x^2 + 4y^2 = 9 \Rightarrow y = \frac{1}{2}\sqrt{9-4x^2}$

$x^2 = 4y$



[1]

Point of intersection

$4 \times 4y + 4y^2 - 9 = 0$

$\Rightarrow (2y - 1)(2y + 9) = 0$

$$y=\frac{1}{2} \quad\Rightarrow\quad x^2=4.\frac{1}{2} \quad\Rightarrow\quad x=\pm\sqrt{2}$$

$y = -\frac{9}{2} \Rightarrow x^2 = -18$ (Not possible) [1]

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{1}{2}\sqrt{9-4x^2} - \frac{x^2}{4}\right] dx$$ [1]

$$= \frac{2}{2}\int_0^{\sqrt{2}} \sqrt{9-4x^2}\,dx - \frac{2}{4}\int_0^{\sqrt{2}} x^2 dx$$

$$= 2\left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2} + \frac{9}{4\times 2}\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^{\sqrt{2}} - \frac{1}{2}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}}$$ [1]

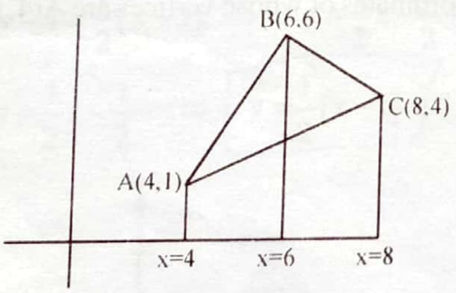
$$= 2\left[\frac{1}{\sqrt{2}}\cdot\frac{1}{2} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right] - \frac{1}{2}\frac{2\sqrt{2}}{3}$$

$$= \frac{1}{\sqrt{2}} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{\sqrt{2}}{3}$$ [1]

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{3} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \text{ sq. units}$$ [1]

OR



[1]

Equation of AB :

$y - 1 = \frac{5}{2}(x-4) \Rightarrow y = \frac{5x}{2} - 9$ [1]

Equation of AC :

$y - 1 = \left(\frac{4-1}{8-4}\right)(x-4) \Rightarrow y = \frac{3x}{4} - 2$ [1]

Equation of BC :

$(y-6) = \left(\frac{4-6}{8-6}\right)(x-6) \Rightarrow y = -x + 12$ [1]

Hence the required area

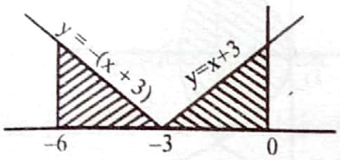
$$= \left|\int_4^6 \left(\frac{5}{2}x - 9 - \frac{3x}{4} + 2\right)dx\right| + \left|\int_6^8 \left(-x + 12 - \frac{3x}{4} + 2\right)dx\right|$$ [1]

$$= \left|\int_4^6 \left(\frac{7x}{4} - 7\right)dx\right| + \left|\int_6^8 \left(-\frac{7x}{4} + 14\right)dx\right|$$

$$= \left|\left[\frac{7x^2}{8} - 7x\right]_4^6\right| + \left|\left[-\frac{7x^2}{8} + 14x\right]_6^8\right| = 7 \text{ sq. units}$$ [1]

7. Using integration, sketch the graph of $y = |x + 3|$ and evaluate the area under the curve $y = |x + 3|$ above x-axis and between $x = -6$ to $x = 0$. **[CBSE 2011, 6M]**

Sol. [1½]



$y = |x + 3|$

$$A = \int_{-6}^{-3} (-x - 3)dx + \int_{-3}^{0} (x + 3)dx$$ [1½]

$$= \left[-\frac{x^2}{2} - 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$$ [1½]

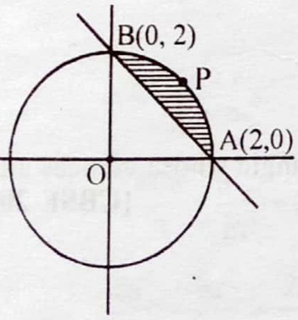
$$= -\frac{9}{2} + 9 + \frac{36}{2} - 18 - \frac{9}{2} + 9 = 9$$ [1½]

8. Using integration, find the area of the region $\{(x, y) : x^2 + y^2 \le 4, x + y \ge 2\}$ **[CBSE 2012, 6M]**

Sol. First we sketch the region whose area is to be found out, this region is the intersection of the following regions

$\{(x,y) : x^2 + y^2 \le 4\}$ and $\{(x, y) : x + y \ge 2\}$ [2]

The point of interection of $x^2 + y^2 = 4$ and $x + y = 2$ are A(2, 0) and B(0, 2)



[1]

Required region is the shaded region APBA

$$\text{Required area} = \int_0^2 (ydx)_{\text{Cirle}} - \int_0^2 (ydx)_{\text{line}}$$

$$\int_0^2 \sqrt{4 - x^2}\,dx - \int_0^2 (2 - x)dx$$ [1]

$$\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2-\left[2x-\frac{x^2}{2}\right]_0^2$$ [1]

$= 0 + 2\frac{\pi}{2} - 2 = (\pi - 2)$ sq.units [1]

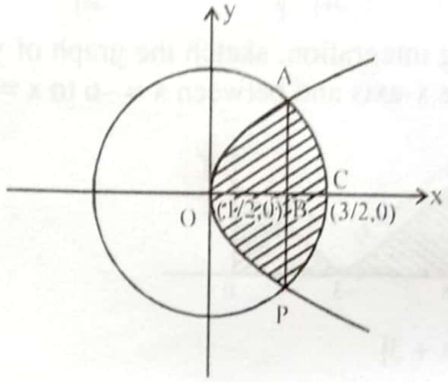
9. Using integration, find the area of the region $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$ using method of integration.

[CBSE 2008, 2013, 6M]

Sol. $y^2 \le 4x,\ 4x^2 + 4y^2 \le 9$

$y^2 = 4x\ \&\ 4x^2 + 4y^2 = 9 \quad \Rightarrow \quad x^2 + y^2 = (3/2)^2$

for intersection point [1]



$4x^2 + 4(4x) = 9$

$4x^2 + 16x - 9 = 0$

$4x^2 + 18x - 2x - 9 = 0$

$2x(2x + 9) - 1(2x + 9) = 0$

$(2x - 1) = 0 \qquad 2x + 9 = 0$

$x = 1/2 \qquad x = -9/2$ [1]

$$\text{Total area} = 2\left[2\int_0^{1/2}\sqrt{x}.dx + \int_{1/2}^{3/2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2dx}\right]$$

$$= 2.\left[\frac{2(x^{3/2})_0^{1/2}}{3/2} + \left(\frac{x}{2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2} + \frac{9}{4\times 2}\sin^{-1}\frac{2x}{3}\right)_{1/2}^{3/2}\right]$$ [1]

$$= 2\left[\left(\frac{4}{3}\times\frac{1}{2\sqrt{2}}\right) + \left(\frac{9}{8}\times\frac{\pi}{2}\right) - \left(\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\frac{1}{3}\right)\right]$$ [½]

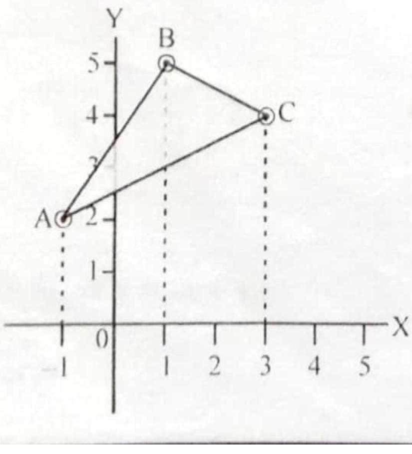
$$= \frac{4}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{\sqrt{2}}{2} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$ [1]

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) = \left[\frac{\sqrt{2}}{6} + \frac{9}{4}\cos^{-1}\frac{1}{3}\right] \text{ sq. units}$$ [½]

10. Using integration, find the area of the region bounded by the triangle whose vertices are (–1, 2), (1, 5) and (3, 4).

[CBSE 2014, 6M]

Sol. [2]



Eq. of line AB is $y=\frac{3}{2}x+\frac{5}{2}$, Eq. of line BC is $y=-\frac{1}{2}x+\frac{7}{2}$ Eq. of line AC is $y=\frac{1}{2}x+\frac{7}{2}$ [1½]

$$|\text{Area of } \Delta ABC| = \left|\int_{-1}^{1}\left(\frac{3}{2}x+\frac{5}{2}\right)dx+\int_{1}^{3}\left(-\frac{1}{2}x+\frac{7}{2}\right)dx-\int_{-1}^{3}\left(\frac{1}{2}x+\frac{7}{2}\right)dx\right| \quad [1]$$

$$=\left|\left(\frac{3}{2}\times\frac{x^2}{2}+\frac{5}{2}x\right)_{-1}^{1}+\left(-\frac{1}{2}\times\frac{x^2}{2}+\frac{7}{2}x\right)_{1}^{3}-\left(\frac{1}{2}\times\frac{x^2}{2}+\frac{7}{2}x\right)_{-1}^{3}\right|$$

$= |5 + 5 - 14| = 4$ square unit [1½]

11. If the area bounded by the parabola $y^2 = 16ax$ and the line $y = 4mx$ is $\frac{a^2}{12}$ sq. units, then using integration, find the value of m. **[CBSE 2015, 6M]**

Sol. Parabola $y^2 = 16ax$

line $y = 4mx$

intersection point of line and parabola

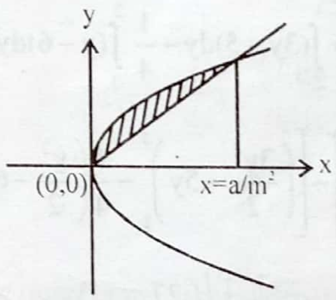
$16m^2x^2 = 16ax$

$\Rightarrow m^2x^2 - ax = 0$

$\Rightarrow x(m^2x-a)=0$

$\Rightarrow x=0;\ x=\frac{a}{m^2}$ [2]

$y=0;\ y=\frac{4a}{m}$



$$\text{Bounded area} = \int_0^{a/m^2}\sqrt{16ax}\,dx-\int_0^{a/m^2}4mx\,dx \quad [1]$$

$$=\frac{2\times4\sqrt{a}}{3}\left[x^{3/2}\right]_0^{a/m^2}-\frac{4m}{2}\left[x^2\right]_0^{a/m^2} \quad [1]$$

$$=\frac{8\sqrt{a}}{3}\times\frac{a^{3/2}}{m^3}-2m\times\frac{a^2}{m^4}$$

$$\Rightarrow \frac{8a^2}{3m^3}-\frac{2a^2}{m^3}=\frac{2a^2}{3m^3} \quad [1]$$

Given bounded area is $\frac{a^2}{12}$

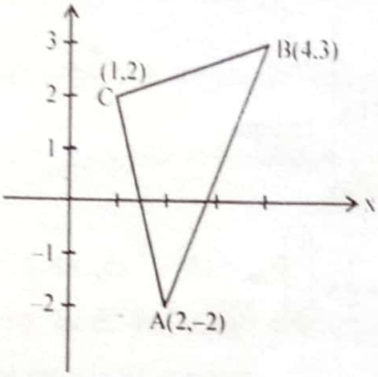
$\therefore \frac{2a^2}{3m^3}=\frac{a^2}{12}$

$m^3 = 8$

$m = 2$ [1]

12. Using the method of integration, find the area of the triangular region whose vertices are (2, –2), (4, 3) and (1, 2). **[CBSE 2015, 6M]**

Sol. [1]



Line AB is : $y = \frac{5}{2}x - 7;\ x = \frac{2}{5}(y+7)$, line BC is $y = \frac{1}{3}(x+5) \Rightarrow x = 3y - 5$

Line AC is ; $y = -4x + 6;\ x = \frac{y-6}{-4}$ [1]

Required area $= \left[\int_{-2}^{3}(\text{lineAB})dy\right] - \left[\int_{2}^{3}(\text{lineBC})dy + \int_{-2}^{2}(\text{lineAC})dy\right]$

$\Rightarrow \left[\frac{2}{5}\int_{-2}^{3}(y+7)dy\right] - \left[\int_{2}^{3}(3y-5)dy - \frac{1}{4}\int_{-2}^{2}(y-6)dy\right]$ [1]

$= \frac{2}{5}\left[\left(\frac{y^2}{2}+7y\right)_{-2}^{3}\right] - \left[\left(\frac{3y^2}{2}-5y\right)_{2}^{3} - \frac{1}{4}\left(\frac{y^2}{2}-6y\right)_{-2}^{2}\right]$ [2]

$= \frac{2}{5}\left[\left(\frac{9}{2}+21\right)-(2-14)\right] - \left[\left\{\left(\frac{27}{2}-15\right)-(6-10)\right\} - \frac{1}{4}\{(2-12)-(2+12)\}\right]$

$= \frac{2}{5}\left[\frac{9}{2}+33\right] - \left[\left(\frac{27}{2}-11\right) - \frac{1}{4}(-24)\right]$

$= \left(\frac{2}{5}\times\frac{75}{2}\right) - \left(\frac{5}{2}+6\right)$

$= 15 - \frac{17}{2} = \frac{13}{2}$ square unit. [1]

NCERT IMPORTANT QUESTIONS	
Examples	7, 9, 10, 13
Exercise # 8.1	4, 6, 9
Exercise # 8.2	5
Miscellaneous Examples	10, 11
Miscellaneous Exercise	15