ROLLE'S THEOREM & LMVT

1. ROLLE'S THEOREM :

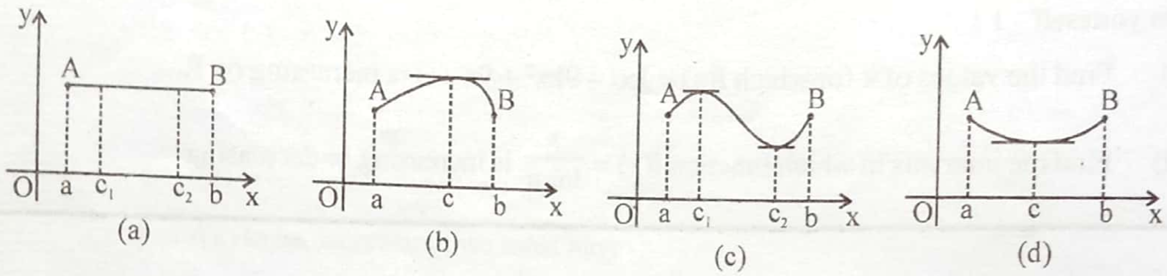
Let f be a function that satisfies the following three conditions :

(a) f is continuous on the closed interval [a, b].

(b) f is differentiable on the open interval (a, b)

(c) f(a) = f(b)

Then there exist at least one number c in (a, b) such that f '(c) = 0.



Note : If f is differentiable function then between any two consecutive roots of f(x) = 0, there is atleast one root of the equation f'(x) = 0.

(d) Geometrical Interpretation :

Geometrically, the Rolle's theorem says that somewhere between A and B the curve has at least one tangent parallel to x-axis.

Illustration 1 : Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2]. **[4]**

Solution : Here we observe that

f(x) is polynomial and since polynomial are always continuous, as well as differentiable. Hence

f(x) is continuous in the [0,2] and differentiable in the (0, 2). [1]

&

$f(0) = 0,\ f(2) = 2^3 - 3.(2)^2 + 2(2) = 0$

$\therefore \quad f(0) = f(2)$ [1]

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exists some c ∈ (0, 2) such that f'(c) = 0

$\Rightarrow \quad f'(c) = 3c^2 - 6c + 2 = 0 \Rightarrow \quad c = 1 \pm \frac{1}{\sqrt{3}}$ [1]

where both $c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$ thus Rolle's theorem is verified. [1]

(b) For f(x) to be decreasing

$f'(x) < 0$

$\Rightarrow (x-2)(x-6) < 0 \quad \Rightarrow x > 2$ and $x < 6$

$\Rightarrow x \in (2, 6)$ [½]

4. Verify Rolle's theorem of the function $f(x) = x^2 - 4x + 3$ on [1, 3]. **[CBSE 2007, 1M]**

Sol. (i) f(x) is continuous in [1, 3] being algebraic function.

(ii) $f'(x) = 2x - 4 \in (1, 3)$

$\therefore$ f(x) is derivable in (1, 3)

(iii) $f(1) = 1^2 - 4.1 + 3 = 1 - 4 + 3 = 0$

$f(3) = 3^2 - 4.3 + 3 = 9 - 12 + 3 = 0$

$\therefore$ f(1) = f(3)

$\therefore$ Rolle's theorem is verified.

Now, there exists a number c such that

$f'(c) = 0$

$\Rightarrow 2c - 4 = 0 \Rightarrow c = 2 \in (1,3)$ [1]

5. The money to the spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when x = 5, and write which value does the question indicate. **[CBSE 2013, 1M]**

Sol. $R(x) = 3x^2 + 36x + 5$

$$\Rightarrow \frac{dR(x)}{dx} = 6x + 36$$

$$\Rightarrow \left[\frac{dR(x)}{dx}\right]_{x=5} = 6\times5+36 = 30+36 = 66.$$ [½]

value based :- more revenue ⇒ more money for the welfare of employees. [Any other individual response with suitable justification be accepted, even if there is no reference to the text]. [½]

Short Answer : [4 Marks]

6. Verify Rolle's Theorem for the following function : $f(x) = \sin x + \cos x,\ x \in \left[0, \frac{\pi}{2}\right]$.

[CBSE 2006, 4M]

Sol. Here $f(x) = \sin x + \cos x,\ x \in \left[0, \frac{\pi}{2}\right]$

Both sinx and cosx are cont. on $\left[0, \frac{\pi}{2}\right]$ and diff.on $\left(0, \frac{\pi}{2}\right)$

$\therefore$ f(x) is cont. on $\left[0, \frac{\pi}{2}\right]$ and

f(x) is diff. on $\left(0, \frac{\pi}{2}\right)$. [1]

Also f(0) = sin0 + cos0 = 1

$f\left(\frac{\pi}{2}\right)=\sin\frac{\pi}{2}+\cos\frac{\pi}{2}=1$

i.e., $f(0)=f\left(\frac{\pi}{2}\right)$ [1]

Thus all the conditions of Rolle's Theorem are satisfied by the function f(x).

$\therefore \quad \exists c\in\left(0,\frac{\pi}{2}\right)$ s.t. $f'(c) = 0$

Now f'(x) = cosx – sinx [1]

$f'(x) = 0 \quad \Rightarrow \cos x = \sin x$

$\Rightarrow \tan x = 1$

$\Rightarrow x=\frac{\pi}{4}$

Clearly $c=\frac{\pi}{4}\in\left(0,\frac{\pi}{2}\right)$ [1]

7. Verify Lagrange's Mean Value Theorem for the following : $f(x) = x^2 + 2x + 3$, [4, 6].

[CBSE 2006, 4M]

Sol. Here $f(x) = x^2 + 2x + 3,\ x \in [4, 6]$

Being a polynomial, $x^2 + 2x + 3$ is both continuous and diff. everywhere and in particular, in [4, 6]. [1]

$f'(x) = 2x + 2$

⇒ The conditions of Lagrange's Mean Value Theorem are satisfied.

$\Rightarrow \exists c\in(4,6)$ satisfying

$f'(c)=\frac{f(b)-f(a)}{b-a} \quad \Rightarrow \quad (2c+2)=\frac{f(6)-f(4)}{6-4}$ [1]

$=\frac{1}{2}[(6^2+2\times6+3)-(4^2+2\times4+3)]$ [1]

$\Rightarrow 2c+2=\frac{1}{2}[36+12-(16+8)]=12$

$\Rightarrow (2c+2) = 10 \Rightarrow c=5$

Clearly $5 \in (4,6)$ [1]

8. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is (i) increasing and (ii) decreasing.

OR

At what points will the tangent to the curve $y = 2x^3 - 15x^2 + 36x - 21$ be-parallel to x-axis? Also, find the equations of tangents to the curve at those points. **[CBSE 2008, 4M]**

Sol. $f(x) = 2x^3 - 9x^2 + 12x + 15$

$f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$= 6(x - 2)(x - 1)$ [1]

For stationary point, $f'(x) = 0$

$6(x-2)(x-1) = 0$

$x = 2,\ x = 1$

$\therefore$ Disjoint intervals are $(-\infty, 1)$ $(1, 2)$, $(2, \infty)$ [1]

Intervals	Test value	Nature of $f'(x)$ $f'(x) = 6(x-2)(x-1)$	Nature of $f(x)$
$(-\infty, 1)$	$x = 0$	$(+)(-)(-) = (+) > 0$	$\uparrow$
$(1, 2)$	$x = 1.5$	$(+)(-)(+) = (-) < 0$	$\downarrow$
$(2, \infty)$	$x = 3$	$(+)(+)(+) = (+) > 0$	$\uparrow$

[1]

$\therefore$ $f(x)$ is increasing in $(-\infty, 1)$, $(2, \infty)$ and decreasing in $(1, 2)$ [1]

OR

$y = 2x^3 - 15x^2 + 36x - 21$

$\frac{dy}{dx} = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$\frac{dy}{dx} = 6(x-3)(x-2)$ [1]

$\because$ Tangent is parallel to x-axis

$\therefore$ $\frac{dy}{dx} = 0$

$6(x-3)(x-2) = 0$

$x = 3,\ x = 2$ [1]

when $x = 3$

$\therefore$ $y = 2(27) - 15(9) + 36(3) - 21 = 6$

$\therefore$ Point $(3, 6)$ [½]

When $x = 2$

$\therefore$ $y = 2(8) - 15(4) + 36(2) - 21 = 7$

$\therefore$ Point $(2, 7)$ [½]

$\therefore$ Required points are $(3, 6)$ and $(2, 7)$

Equation of tangent at $(3, 6)$ is

$y - y_1 = m(x - x_1)$

$y - 6 = 0(x - 3)$

$y - 6 = 0$ [½]

Equation of tangent at $(2, 7)$ is

$y - 7 = 0(x - 2)$

$y - 7 = 0$

[½]

9. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8 cm and y = 6 cm, find the rate of change of (a) the perimeter (b) the area of the rectangle

[CBSE 2009, 4M]

OR

Find the intervals in which the function f given by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

Sol. Given : $\frac{dx}{dt} = -5$ cm/min

$\frac{dy}{dt} = 4$ cm/min

Perimeter $P = 2(x + y)$

$$\frac{dP}{dt} = \frac{2dx}{dt} + \frac{2dy}{dt}$$

$$= 2(-5) + 2(4) = -2 \text{ cm/min}$$ [2]

Area $A = xy$

$$\frac{dA}{dt} = \frac{xdy}{dt} + y\frac{dx}{dt}$$

$= 8 \times 4 + 6(-5) = 2\text{cm}^2/\text{min}$ [2]

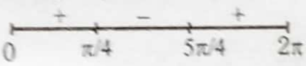
OR

$f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

$f'(x) = 0 \Rightarrow \sin x = \cos x$

$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $(0 \le x < 2\pi)$ [1]



$f'(x) > 0$ if $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ [1½]

i.e. f is strictly increasing in the interval

$\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

Also $f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

i.e. f is strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ [1½]

10. Find the equations of the tangent and the normal to the curve $x = 1 - \cos\theta$, $y = \theta - \sin\theta$ at $\theta = \frac{\pi}{4}$.

[CBSE 2010, 4M]

Sol. $x = 1 - \cos\theta$, $y = \theta - \sin\theta$, at $\theta = \frac{\pi}{4}$

$x_1 = 1 - \frac{1}{\sqrt{2}}$ $\qquad y_1 = \frac{\pi}{4} - \frac{1}{\sqrt{2}}$ [1]

$\frac{dx}{d\theta} = \sin\theta$, $\qquad \frac{dy}{d\theta} = 1 - \cos\theta$

$$\frac{dy}{dx} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2} - 1$$ [1]

Equation of tangent at (x_1, y_1)

$$\left(y-\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)=\left(\sqrt{2}-1\right)\left(x-\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$ **[1]**

Equation of normal at (x_1, y_1)

$$\left(y-\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)=-\left(\sqrt{2}+1\right)\left(x-1+\frac{1}{\sqrt{2}}\right)$$ **[1]**

11. Prove that $y=\dfrac{4\sin\theta}{(2+\cos\theta)}-\theta$ is an increasing function in $\left[0,\dfrac{\pi}{2}\right]$ **[CBSE 2011, 4M]**

OR

If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area. **[CBSE 2011, 4M]**

Sol. $y=\dfrac{4\sin\theta}{(2+\cos\theta)}-\theta$

$$y'=\frac{4(2+\cos\theta)\cos\theta-4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2}-1$$

$$=\frac{4[2\cos\theta+\cos^2\theta+\sin^2\theta]}{(2+\cos\theta)^2}-1$$

$$=\frac{4(1+2\cos\theta)}{(2+\cos\theta)^2}-1=\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}\quad ...(1)$$ **[2]**

$\because\ \cos\theta\geq 0$ if $\theta\in\left[0,\dfrac{\pi}{2}\right]$

$4-\cos\theta>0$ if $\theta\in\left[0,\dfrac{\pi}{2}\right]$

$(2+\cos\theta)^2>0$ if $\theta\in\left[0,\dfrac{\pi}{2}\right]$

$\therefore\ \ y'>0$ in $\left[0,\dfrac{\pi}{2}\right]$

$\Rightarrow$ y is increasing function. ...(2) **[2]**

OR

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then, $r=9$ cm and $\Delta r=0.03$ cm

Let V be the volume of the sphere. Then,

$$V=\frac{4}{3}\pi r^3\ \Rightarrow\ \frac{dV}{dr}=4\pi r^2.$$

$$\Rightarrow\left(\frac{dV}{dr}\right)_{r=9}=4\pi\times 9^2=324\pi$$ **[2]**

Let ΔV be the error in V due to error Δr in r. Then,

$$\Delta V = \frac{dV}{dr}\Delta r$$

$\Rightarrow \Delta V = 324\pi \times 0.03 = 9.72\,\pi\ cm^3$. [2]

Thus, the approximate error in calculating the volume is $9.72\,\pi\ cm^3$.

12. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ? **[CBSE 2012, 4M]**

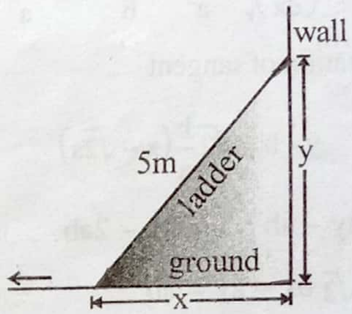
Sol. If the foot of the ladder is at a distance x from the wall and the top is at a height y at instant of time t, then

$(5)^2 = x^2 + y^2$ [½]

Differentiating w.r.t. time (t)

$$\Rightarrow \frac{d}{dt}(25) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad ....(1)$$ [½]

We are given that $\frac{dx}{dt} = 0.02$ m/sec,



$x = 4$ m and $y = \sqrt{25-4^2}$ m = 3 m

$(\because x^2 + y^2 = 25\ m^2,\ y = \sqrt{25-x^2}\ m)$ [1]

Hence from (1)

$$0 = 2 \times 4\ m \times 0.02\ m/sec + 2 \times 3\ m\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = -\frac{0.16}{6}\ m/sec.$$ [1]

$\therefore$ Rate of decrease of height of ladder on the wall

$$= \frac{16}{600}\ m/sec = \frac{1600}{600}\ cm/sec = \frac{8}{3}\ cm/sec.$$ [1]

13. Find the value(s) of x for which $y = [x(x-2)]^2$ is an increasing function. **[CBSE 2014, 4M]**

OR

Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}\,a, b)$.

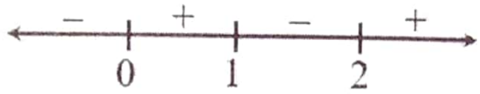
Sol. $y = [x(x-2)]^2$ = 2[x(x−2)]×[2x−2]
4x(x−2)(x−1)

$$\frac{dy}{dx} = 4x(x-1)(x-2)$$ [1]

For increasing function

$$\frac{dy}{dx} > 0$$

$4x(x-1)(x-2) > 0$ **[1½]**



$\therefore (0, 1) \cup (2, \infty)$ **[1½]**

OR

Curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{b^2}{a^2}\left(\frac{x}{y}\right)$$

Slope of tangent at $P(\sqrt{2}\,a, b)$

$$\left(\frac{dy}{dx}\right)_P = \frac{b^2}{a^2} \times \frac{\sqrt{2}a}{b} = \sqrt{2}\frac{b}{a}$$ **[1]**

Equation of tangent

$$y - b = \sqrt{2}\frac{b}{a}(x - \sqrt{2}a)$$

$\Rightarrow ay - ab = \sqrt{2}\,bx - 2ab$

$\Rightarrow \sqrt{2}\,bx - ay = ab$ **[1½]**

Equation of Normal

$$y - b = \frac{-a}{\sqrt{2}b}(x - \sqrt{2}a)$$

$$\sqrt{2}by - \sqrt{2}b^2 = -ax + \sqrt{2}a^2$$

$$ax + \sqrt{2}by = \sqrt{2}(a^2 + b^2)$$ **[1½]**

14. Find the equation of tangents to the curve $y = x^3 + 2x - 4$, which are perpendicular to line $x + 14y + 3 = 0$. **[CBSE 2016, 4M]**

Sol. Curve $y = x^3 + 2x - 4$

Slope of tangent $m_1 = \frac{dy}{dx} = 3x^2 + 2$ ......(1)

Slope of line $m_2 = \frac{-1}{14}$ ......(2)

Tangent is $\perp$ to the line

$\therefore \quad m_1 \times m_2 = -1$

$(3x^2 + 2) \times \frac{-1}{14} = -1$ **[1]**

$3x^2 = 12, \ x = \pm 2$ put in equation (1) point of contact are $P_1(2, 8)$, $P_2(-2, -16)$ [1]

equation of tangent of $P_1(2, 8)$

$y - 8 = 14(x - 2)$

$14x - y - 20 = 0$ [1]

Equation of tangent at $P_2(-2, -16)$

$y + 16 = 14(x + 2)$

$14x - y + 12 = 0$ [1]

Long Answer : [6 Marks]

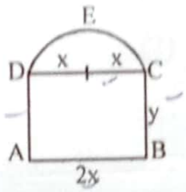
15. A window is in the form of a rectangle surmounted by a semi-circle. If the total perimeter of the window is 30m, find the dimensions of the window so that maximum light is admitted.

[CBSE 2000, 2002, 2006, 6M]

Sol. Let AB = 2x, BC = y

$\Rightarrow$ Radius r of the semicircle = x

$\therefore$ According to the question



$2x + y + \pi x + y = 30$

$$\Rightarrow y = 15 - \frac{\pi + 2}{2}x \quad .....(1)$$ [1]

Let A be the area of the window. Then

$$A = 2x \times y + \frac{1}{2}\pi x^2 = 2x\left(15 - \frac{\pi+2}{2}x\right) + \frac{1}{2}\pi x^2$$

$$= 30x - (\pi + 2)x^2 + \frac{1}{2}\pi x^2 = 30x - \left(\frac{\pi}{2} + 2\right)x^2$$ [1]

$$\Rightarrow \frac{dA}{dx} = 30 - (\pi + 4)x \text{ and } \frac{d^2A}{dx^2} = -(\pi + 4) \quad .....(2)$$ [½ + ½]

Now for max. or min.,

$$\frac{dA}{dx} = 0 \Rightarrow x = \frac{30}{\pi + 4} m$$ [1]

For this value of x, clearly $\frac{d^2A}{dx^2} < 0$

$\therefore$ Maximum light will come through the window. [½]

$$\text{Now } AB = 2x = \frac{60}{\pi+4} m; r = x = \frac{30}{\pi+4} m$$ [½]

$$BC = y = 15 - \frac{\pi+2}{2}x \quad \text{[From (1)]}$$

$$= 15 - \frac{\pi+2}{2} \times \frac{30}{\pi+4} = \frac{15(\pi + 4 - \pi - 2)}{\pi + 4} = \frac{30}{\pi+4} m$$ [1]

16. Find the point on the curve $x^2 = 4y$ which is nearest to the point $(-1, 2)$. **[CBSE 2007, 6M]**

OR

A wire of length 28 m is to be cut into two pieces. One of the two pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that that the combined area of these is minimum ?

Sol. Let A(h,k) be any point on the curve $x^2 = 4y$ and B(–1,2) be the given point

$\therefore AB = \sqrt{(h+1)^2 + (k-2)^2}$ ...(1)

$\because$ point A(h,k) lies on the curve

$\therefore h^2 = 4k \Rightarrow k = \frac{h^2}{4}$ ...(2) [1]

Putting k in equation (1)

$$AB = \sqrt{(h+1)^2 + \left(\frac{h^2}{4} - 2\right)^2}$$

Squaring both sides, we get

$$AB^2 = (h+1)^2 + \left(\frac{h^2}{4} - 2\right)^2$$

Let $AB^2 = f(h)$

$\therefore f(h) = (h+1)^2 + \left(\frac{h^2}{4} - 2\right)^2$ ....(3) [1½]

$$f'(h) = 2(h+1) + 2\left(\frac{h^2}{4} - 2\right) \cdot \frac{2h}{4}$$

$$f'(h) = 2h + 2 + \frac{h^3}{4} - 2h = 2 + \frac{h^3}{4}$$

For stationary point, $f'(h) = 0$

$\Rightarrow 2 + \frac{h^3}{4} = 0 \Rightarrow h^3 = -8 \Rightarrow h = -2$. [1½]

Now, $f''(h) = \frac{3h^2}{4}$

At $h = -2$, we have

$$f''(-2) = \frac{3(-2)^2}{4} = \frac{3.4}{4} = 3 > 0$$

$\therefore$ f(h) is min. at $h = -2$

$\therefore$ Distance is max. at $h = -2$

Putting $h = -2$ in equation (2), we get [1]

$$K = \frac{-2^2}{4} = \frac{4}{4} = 1$$

$\therefore$ Required point $(h, k) = (-2, 1)$ [1]

OR

Let x be the length of the pieces to be made into circle.

$\therefore$ (28 – x) be the length of the piece to be made into square.

$$2\pi r = x \Rightarrow r = \frac{x}{2\pi}$$

Area of circle $A_1 = \pi r^2 = \pi.\frac{x^2}{4\pi^2} = \frac{x^2}{4\pi}$ ...(1) [1]

Perimeter of square = 28–x

$$4\ell = 28 - x \Rightarrow \ell = \frac{28-x}{4}$$

Area of square $A_2 = \ell^2 = \frac{(28-x)^2}{16}$ ....(2) [1]

Combined area $A = A_1 + A_2$

$$A = \frac{x^2}{4\pi} + \frac{(28-x)^2}{16}$$

$$\frac{dA}{dx} = \frac{2x}{4\pi} + \frac{2(28-x)}{16}(-1)$$

$$\frac{dA}{dx} = \frac{x}{2\pi} - \frac{28-x}{8}$$ [1]

For stationary point, $\frac{dA}{dx} = 0$

$$\Rightarrow \frac{x}{2\pi} - \frac{28-x}{8} = 0 \Rightarrow \frac{4x - 28\pi + \pi x}{8\pi} = 0$$

$$\Rightarrow x(\pi + 4) - 28\pi = 0 \Rightarrow x = \frac{28\pi}{\pi+4}$$ [1]

Now, $\frac{d^2A}{dx^2} = \frac{1}{2\pi} + \frac{1}{8} > 0$

$\therefore$ Area is min. at $x = \frac{28\pi}{\pi+4}$ [1]

$\therefore$ Length of I[st] piece $\Rightarrow x = \frac{28\pi}{\pi+4}$

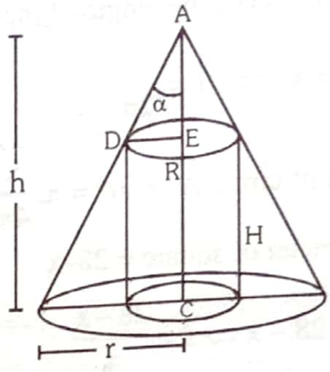
II[nd] piece $= 28 - x = 28 - \frac{28\pi}{\pi+4}$

$$= \frac{28\pi + 112 - 28\pi}{\pi+4} = \frac{112}{\pi+4}$$

$\therefore$ Required length of pieces

$= \frac{28\pi}{\pi+4}$ cm, $\frac{112}{\pi+4}$ cm. [1]

17. Show that the volume of the greatest cyllinder that can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$. **[CBSE 2008,10, 6M]**



Sol. Let radius of cylinder = R

height of cylinder = H

Volume of cylinder $V = \pi R^2 H$ ..... (1)

$AE = AC - EC = h - H$

$\therefore \quad \tan\alpha = \frac{DE}{AE}$

$\tan\alpha = \frac{R}{h-H} \Rightarrow R = (h-H)\tan\alpha$ ..... (2)

Putting the value of R in eq. (1), we get

$\therefore \quad V = \pi(h-H)^2\tan^2\alpha.H$

$V = \pi\tan^2\alpha(h-H)^2H$ ...... (3) [1]

Diff. w.r.t. H, we get

$$\frac{dV}{dH} = \pi\tan^2\alpha[(h-H)^2.1 + H.2(h-H)(-1)]$$

$= \pi\tan^2\alpha(h-H)[h-H-2H]$

$= \pi\tan^2\alpha(h-H)(h-3H)$ ..... (4) [1]

For stationary point, $\frac{dV}{dH} = 0$

$\pi\tan^2\alpha(h-H)(h-3H) = 0$

$h - H = 0$ or $h - 3H = 0$

$\therefore \quad H = h$ or $H = \frac{h}{3}$

$\because \quad H = h$ is not possible

$\therefore \quad H = \frac{h}{3}$ [1]

Now $\frac{d^2V}{dH^2} = \pi\tan^2\alpha[(h-H)(-3) + (h-3H)(-1)]$ [1]

$= \pi\tan^2\alpha[-3h + 3H - h + 3H]$

$= \pi\tan^2\alpha[6H - 4h]$

$\therefore \quad \left(\frac{d^2V}{dH^2}\right)_{\text{at } H=\frac{h}{3}} = \pi\tan^2\alpha\left(6.\frac{h}{3} - 4H\right)$

$= \pi\tan^2\alpha(2h - 4h)$

$= \pi\tan^2\alpha(-2h) < 0$

$\therefore$ Volume is maximum at $H = \frac{h}{3}$ [1]

Putting H in equation (3),

Maximum volume $= \pi \tan^2\alpha \left(h - \frac{h}{3}\right)^2 . \frac{h}{3}$

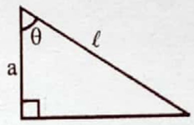
$= \pi \tan^2\alpha . \frac{4h^2}{9} . \frac{h}{3} = \frac{4}{27} \pi h^3 \tan^2\alpha$ [1]

18. If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$ **[CBSE 2009, 6M]**

OR

A manufacturer can sell x items at a price of Rs. $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is Rs. $\left(\frac{x}{5} - 100\right)$. Find the number of items he should sell to earn maximum profit.

Sol. $A = \frac{1}{2} \ell \cos\theta . \ell \sin\theta$



$= \frac{1}{2} \ell^2 \sin\theta \cos\theta$ [1]

$= \frac{1}{2}\left(\frac{S}{1+\cos\theta}\right)^2 . \sin\theta \cos\theta, [\ell\cos\theta + \ell = S]$

$A = \frac{S^2}{4} \frac{\sin 2\theta}{(1+\cos\theta)^2}$ [1]

$\frac{dA}{d\theta} = \frac{S^2}{4}\left(\frac{(1+\cos\theta)^2 2\cos 2\theta - 2\sin 2\theta(-\sin\theta)(1+\cos\theta)}{(1+\cos\theta)^4}\right) = 0$ [1½]

$\Rightarrow \quad (1+\cos\theta)(\cos 2\theta + \cos\theta) = 0$

$\Rightarrow \quad (1+\cos\theta)(2\cos^2\theta + \cos\theta - 1) = 0$

$\Rightarrow \quad (1+\cos\theta)(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\frac{dA}{d\theta} = (1+\cos\theta)^2(2\cos\theta - 1)$ [1½]

$\left.\begin{array}{l}\text{If } \theta < \pi/3 \Rightarrow \frac{dA}{d\theta} > 0 \\ \text{If } \theta > \pi/3 \Rightarrow \frac{dA}{d\theta} < 0\end{array}\right\} \Rightarrow \theta = \frac{\pi}{3}$ is pt. of maxima [1]

OR

Let P be the profit = S.P. – C.P. [1]

$$P = x\left(5-\frac{x}{100}\right)-\left(\frac{x}{5}+500\right)$$ [1]

$$P = \frac{24x}{5}-\frac{x^2}{100}-500$$ [1]

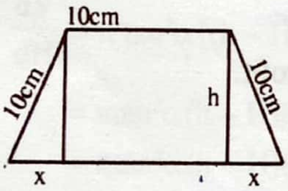
$$\frac{dP}{dx} = \frac{24}{5}-\frac{x}{50} = 0 \Rightarrow x = 240$$ [1]

$$\frac{d^2P}{dx^2} = -\frac{1}{50} = \text{negative}$$ [1]

$\therefore$ profit is maximum at $x = 240$. [1]

19. If the length of three sides of a trapezium other than the base is 10 cm each, find the area of the trapezium, when it is maximum. **[CBSE 2010, 6M]**

Sol.



$h = \sqrt{100-x^2}$ [1]

$\therefore$ Area of the trapezium

$= \frac{1}{2}$(sum of parallel sides) $\times$ height

$$A = \frac{1}{2}\times(10+10+2x)\times\sqrt{100-x^2}$$ [1]

$$\frac{dA}{dx} = \frac{1}{2}\left[2\sqrt{100-x^2}+\frac{(20+2x)(-2x)}{2\sqrt{100-x^2}}\right]$$ [1]

for extreme values, $\frac{dA}{dx} = 0$

$$\frac{2(100-x^2)-20x-2x^2}{2\sqrt{100-x^2}} = 0$$

$\Rightarrow \quad 100 - x^2 - 10x - x^2 = 0$

$\Rightarrow \quad x^2 + 5x - 50 = 0$

$\Rightarrow \quad (x + 10)(x - 5) = 0$

$\Rightarrow \quad x = 5, x = -10$ (Reject) [1]

about $x = 5$

$$\frac{d^2A}{dx^2} = -\frac{30}{\sqrt{75}} < 0$$ [1]

so, x = 5 is the point of maximum.

$$A=\frac{1}{2}(10+10+2\times5)\times5\sqrt{3}=75\sqrt{3} \text{ sq. units}$$ [1]

20. Show that the right-circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. **[CBSE 2011, 6M]**

OR

A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12m, find the dimensions of the rectangle that will produce the largest area of the window.

Sol. Let r be radius, ℓ be slant height, h be height of cone of given volume

$$V=\frac{1}{3}\pi r^2 h \Rightarrow r^2=\frac{3V}{\pi h}$$

Let S be curved surface area

$$S=\pi r\ell=\pi r\sqrt{r^2+h^2}$$ [1]

$$S^2=\pi^2 r^2(r^2+h^2)=\frac{\pi^2 3V}{\pi h}\left[\frac{3v}{\pi h}+h^2\right]$$

$$S^2=3V\left[\frac{3V}{\pi h^2}+h\right]\pi$$ [1]

Let $S^2 = z$ then S is max. or min. according z is max. or min.

$$z=3V\left[\frac{3V}{\pi h^2}+h\right]\pi$$

$$\frac{dz}{dh}=3V\left[\frac{-6V}{\pi h^3}+1\right]\pi$$ [1]

for max. or min. $\frac{dz}{dh}=0 \Rightarrow V=\frac{\pi h^3}{6}$ [1]

$$\frac{d^2z}{dh^2}=3\pi V\left[\frac{18V}{\pi h^4}\right]>0$$ [1]

Hence z is minimum when $V=\frac{\pi h^3}{6}$

Now $\frac{\pi h^3}{6}=\frac{\pi r^2 h}{3} \Rightarrow h^2=2r^2$

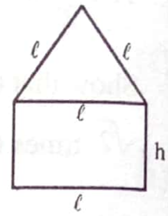
$\Rightarrow h=\sqrt{2}r$ [1]

OR

Let length of rectangle l & breadth is h

$P = 4\ell + 2h = 12 \Rightarrow 2\ell + h = 6$

$\Rightarrow h = 6 - 2\ell$ [1]



$A = \ell h + \frac{\sqrt{3}}{4}\ell^2$

$A = \ell(6-2\ell) + \frac{\sqrt{3}}{4}\ell^2$ [1]

$\frac{dA}{d\ell} = 6 - 4\ell + \frac{\sqrt{3}}{4}2\ell$ [1]

$\frac{d^2A}{d\ell^2} = -4 + \frac{\sqrt{3}}{2} < 0$ [1]

$\therefore$ Area of window is max. when

$\frac{dA}{d\ell} = 0 \Rightarrow \ell = \frac{12}{8-\sqrt{3}}$ [1]

$h = 6 - \frac{24}{8-\sqrt{3}}$ [1]

21. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. **[CBSE 2012, 6M]**

OR

An Open box with a square base is to be made out of a given quantity of cardboard of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

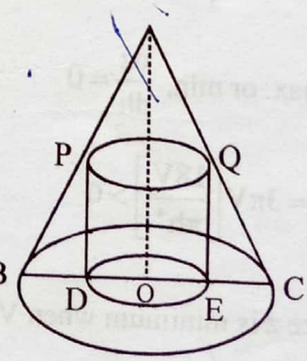
[CBSE 2005, 2006, 2012, 6M]

Sol. Let OC = r be the radius of the cone and OA = h be its height. Let a cylinder with radius OE = x inscribed in the given cone. The height QE of the cylinder is given by [1]

$\frac{QE}{OA} = \frac{EC}{OC}$ (since $\Delta QEC \sim \Delta AOC$)

or $\frac{QE}{h} = \frac{r-x}{r}$

or $QE = \frac{h(r-x)}{r}$ [1]



Let S be the curved surface area of the given cylinder. Then

$S \equiv S(x) = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$ [1]

or $\begin{cases} S'(x) = \frac{2\pi h}{r}(r-2x) \\ S''(x) = \frac{-4\pi h}{r} \end{cases}$ [1 + 1]

Now $S'(x) = 0$ gives $x = \frac{r}{2}$. since $S''(x) < 0$ for all x, $S''\left(\frac{r}{2}\right) < 0$. So $x = \frac{r}{2}$ is a point of maxima of S. Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. [1]

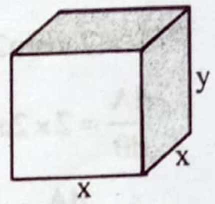
OR

Let each side of the square base be x and height by y then

$$c^2 = \text{total surface area of the box} = 4xy + x^2 \quad \text{..... (i)}$$ [1]

Let V be the corresponding volume of the box, then

Volume of open box = Area of base × height



$$V = x^2y = x^2\left(\frac{c^2 - x^2}{4x}\right)$$

or $$V = \frac{1}{4}(c^2x - x^3),\ 0 < x < c \quad \text{..... (ii)}$$ [1]

$$\frac{dv}{dx} = \frac{1}{4}(c^2 - 3x^2)$$ [½]

$$\frac{d^2v}{dx^2} = \frac{1}{4} \times -6x = -\frac{3}{2}x$$ [½]

For maximum value of volume

$$\frac{dv}{dx} = \frac{1}{4}(c^2 - 3x^2) = 0$$

$$\Rightarrow \quad c^2 - 3x^2 = 0 \Rightarrow x = \frac{c}{\sqrt{3}}$$ [1]

$$\left(\frac{d^2v}{dx^2}\right)_{x=\frac{c}{\sqrt{3}}} = -\frac{3}{2} \times \frac{c}{\sqrt{3}} < 0$$

Therefore V is maximum at $x = \frac{c}{\sqrt{3}}$ and maximum volume will be [1]

$$V = \frac{1}{4}\left(c^2 \times \frac{c}{\sqrt{3}} - \frac{c^3}{3\sqrt{3}}\right) = \frac{1}{4}\left(\frac{2c^3}{3\sqrt{3}}\right)$$

$$V = \frac{c^3}{6\sqrt{3}} \text{ cubic units}$$ [1]

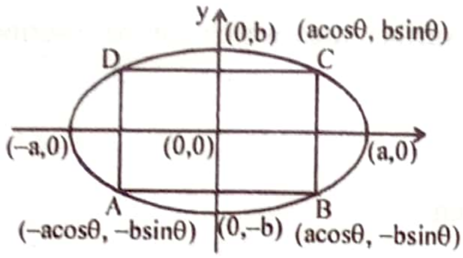
22. Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

[CBSE 2013, 6M]

OR

Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which pass through the point $\left(\frac{4}{3}, 0\right)$.

Sol. Let ABCD is rectangle that can be inscribed in an ellipse



[1]

Area of rectangle (A) = (AB) × (BC) = (2acosθ)(2bsinθ)

$A = 2ab\sin 2\theta$ [1]

$\dfrac{dA}{d\theta} = 2 \times 2ab.\cos 2\theta$ [1]

for critical point $\dfrac{dA}{d\theta} = 0$ [1]

$\cos 2\theta = 0$

$\cos 2\theta = \cos\dfrac{\pi}{2}$

$\theta = \dfrac{\pi}{4}$

for maximum $\dfrac{d^2A}{d\theta^2} = -8ab\sin 2\theta$

$\left(\dfrac{d^2A}{d\theta^2}\right)_{\text{at } \theta=\frac{\pi}{4}} < 0$ [1]

So area of maximum at $\theta = \dfrac{\pi}{4}$

Thus area is ABCD $= 2ab.\sin 2\times\dfrac{\pi}{4}$ [1]

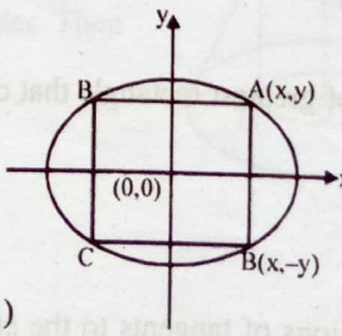
$\Rightarrow \quad 2ab.\sin\dfrac{\pi}{2} = 2ab$

[Aliter]

Given $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ....(1)

Let point A be (x,y) for fig. AB = 2y and BC = 2x. [1]

∴ Area of ABCD is



Let $A = (AB) \times (BC)$

$A = 4xy \Rightarrow A^2 = 16x^2y^2$ [1]

Let $S = A^2 = 16x^2 \times \dfrac{b^2}{a^2}(a^2 - x^2)$ from equation (1)

$$S = \frac{16b^2}{a^2}[x^2.(a^2 - x^2)] = \frac{16b^2}{a^2}\left(a^2x^2 - x^4\right)$$ [1]

$$\frac{ds}{dx} = \frac{16b^2}{a^2}\left[2a^2x - 4x^3\right]$$

put $\frac{ds}{dx} = 0 \Rightarrow x = \frac{a}{\sqrt{2}}$ [1]

$$\frac{d^2s}{dx^2} = \frac{16b^2}{a^2}\left[2a^2 - 12x^2\right]$$

$$\left(\frac{d^2s}{dx^2}\right)_{x=\frac{a}{\sqrt{2}}} = \frac{16b^2}{a^2}\left[2a^2 - 6a^2\right] < 0$$ [1]

for maximum area $x = \frac{a}{\sqrt{2}}$ and $y = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{2}} = \frac{b}{\sqrt{2}}$ [½]

Hence maximum are $= 4xy = 4 - \frac{a}{\sqrt{2}}.\frac{b}{\sqrt{2}} = 2ab$ [½]

OR

Let $P(x_1, y_1)$ be the point in the curve so equation of curve $3x^2 - y^2 = 8$

slope of tangent $6x - 2y\frac{dy}{dx} = 0$

$$\left(\frac{dy}{dx}\right) = \frac{6x}{2y}$$

$$\left(\frac{dy}{dx}\right)_P = 3\left(\frac{x_1}{y_1}\right) \quad ......(1)$$ [1]

Equation of tangent $y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$

$$y - y_1 = 3\left(\frac{x_1}{y_1}\right)(x - x_1) \quad ......(2)$$ [1]

Equation (2) passes through the point $\left(\frac{4}{3}, 0\right)$

$$0 - y_1 = 3\left(\frac{x_1}{y_1}\right)\left(\frac{4}{3} - x_1\right)$$

$$-y_1 = 3\left(\frac{x_1}{y_1}\right)\left(\frac{4 - 3x_1}{3}\right)$$

$$-y_1^2 = 4x_1 - 3x_1^2$$

$$8 - 3x_1^2 = 4x_1 - 3x_1^2 \Rightarrow x_1 = 2$$ [1]

Now $y_1^2 = 3x_1^2 - 8$

$y_1^2 = 12 - 8$

$y_1^2 = 4$

$y_1 = \pm 2$

Points are (2, 2) & (2, –2) [1]

Equation of tangent at (2, 2)

$$(y - 2) = 3\left(\frac{2}{2}\right)(x - 2)$$

$y - 2 = 3x - 6$

$3x - y = 4$ [1]

equation of tangent at (2, –2)

$$(y + 2) = 3\left(\frac{2}{-2}\right)(x - 2)$$

$y + 2 = -3(x - 2)$

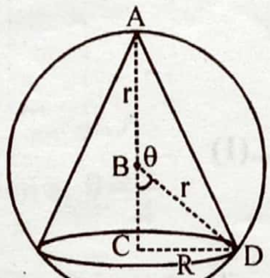
$y + 2 = -3x + 6$

$3x + y = 4$ [1]

23. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

[CBSE 2014, 6M, Set-I]

Sol. Let R and h be the radius and height of the cone respectively.



[1]

The volume (V) of the cone is given by

$$V = \frac{1}{3}\pi R^2 h$$

$r^2 = (h - r)^2 + R^2$

$r^2 = h^2 + r^2 - 2hr + R^2$

$R^2 = 2hr - h^2$ ...(i) [1]

Now Volume of cone $V = \frac{1}{3}\pi R^2 h$

$$V = \frac{\pi h}{3}(2hr - h^2) \Rightarrow \frac{\pi}{3}(2h^2 r - h^3)$$

$$\Rightarrow \frac{dv}{dh} = \frac{\pi}{3}(4hr - 3h^2) = 0$$

$$\Rightarrow h = \frac{4r}{3}$$ [1½]

and $\frac{d^2v}{dh^2} = \frac{\pi}{3}(4r - 6h)$

$\frac{d^2v}{dh^2} < 0$ at $h = \frac{4r}{3}$ [1]

v is max at $h = \frac{4r}{3}$

max. vol. of cone $= \frac{1}{3}\pi R^2 h$

$= \frac{\pi}{3}(2hr - h^2)h$

$= \frac{\pi}{3}(2h^2 r - h^3)$

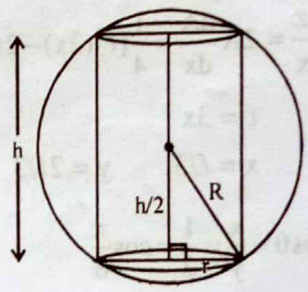
putting the value of h = 4r/3

$\Rightarrow \frac{\pi}{3}\left[2r\left(\frac{4r}{3}\right)^2 - \left(\frac{4r}{3}\right)^3\right] = \frac{4}{3}\pi r^3\left[\frac{8}{27}\right]$

$\Rightarrow \frac{8}{27}$ (Vol. of Sphere) [1½]

24. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. **[CBSE 2014, 6M, Set-II]**

Sol. Given R be the radius of sphere.



[½]

Let h is height and r is radius of cylinder

$R^2 = \frac{h^2}{4} + r^2$

$r^2 = R^2 - \frac{h^2}{4} \Rightarrow \frac{4R^2 - h^2}{4}$ ....(i)

Vol. of cylinder $V = \pi r^2 h = \frac{\pi h}{4}(4R^2 - h^2) = \pi R^2 h - \frac{\pi}{4}h^3$ ...(ii) [1½]

$\frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4} = 0$

$\Rightarrow \pi R^2 = \frac{3\pi}{4}h^2 \Rightarrow h^2 = \frac{4R^2}{3}$

$\Rightarrow h = \frac{2R}{\sqrt{3}}$ [1]

Now $\frac{d^2V}{dh^2} = -\frac{6\pi h}{4} \Rightarrow \frac{d^2V}{dh^2} = \frac{-6\pi}{4} \times \frac{2R}{\sqrt{3}} = -\sqrt{3}\pi R$

$\Rightarrow \frac{d^2V}{dh^2} < 0$ at $h = \frac{2R}{\sqrt{3}}$

Hence Vol. is maximum at $h = \frac{2R}{\sqrt{3}}$ [1]

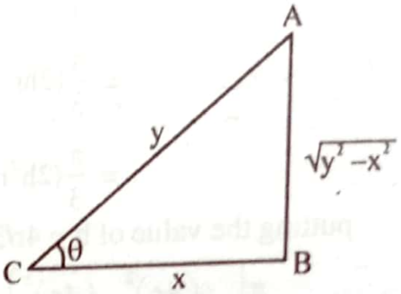
$$V = \pi R^2 \times \frac{2R}{\sqrt{3}} - \frac{\pi}{4}\left(\frac{2R}{\sqrt{3}}\right)^3 \qquad \text{from Eq. (ii)}$$

$$V = \frac{4\pi R^3}{3\sqrt{3}}$$ [2]

25. If the sum of lengths of the hypotenuse and a side of a right triangle is given, show that the area of the triangle is maximum, when the angle between them is 60°. **[CBSE 2014, 6M, Set-II]**

Sol. Let y be the length of hypotenuse

and x be the length of base



Given $\ell = x + y$ [1]

$$A = \frac{1}{2} \times x \times \sqrt{y^2 - x^2}$$

Let $(z) = A^2 = \frac{x^2}{4}(y^2 - x^2)$

$$z = A^2 = \frac{x^2}{4}[\ell^2 + x^2 - 2\ell x - x^2]$$

$$z = A^2 = \frac{1}{4}[\ell^2 x^2 - 2\ell x^3]$$ [1]

$$\frac{dz}{dx} = 2A\frac{dA}{dx} = \frac{1}{4}[\ell^2(2x) - 2\ell(3x^2)] = 0$$ [1]

$$\ell = 3x$$

$$x = \ell/3, \qquad y = 2\ell/3$$ [1½]

$$\cos\theta = \frac{x}{y} = \frac{1}{2} = \cos\frac{\pi}{6}$$

$$\theta = \frac{\pi}{6}$$

Now $\frac{d^2z}{dx^2} = \frac{1}{2}\left[\ell^2 - 6\ell x\right]$

$$\left(\frac{d^2z}{dx^2}\right)_{x=\frac{\ell}{3}} = \frac{1}{2}\left[\ell^2 - 2\ell^2\right]$$

$$= -\frac{\ell^2}{2} < 0$$

Thus area of triangle is maximum at $\theta = \pi/6$ [1½]

26. Tangent to the circle $x^2 + y^2 = 4$ at any point on it in the first quadrant makes intercepts OA and OB on x and y axes respectively, O being the centre of the circle. Find the minimum value of (OA + OB). **[CBSE 2015, 6M]**

Sol. Circle $x^2 + y^2 = 4$

In Δ OPA

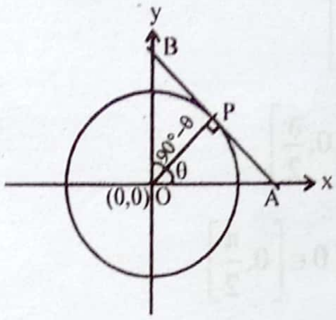
$$\cos\theta = \frac{2}{OA}$$

$$OA = 2\sec\theta$$

In ΔOPB

$$\cos(90-\theta) = \frac{2}{OB}$$

$$OB = 2\text{cosec}\theta.$$



(let) $Z = OA + OB$

$$Z = 2(\sec\theta + \text{cosec}\theta)$$

$$\frac{dZ}{d\theta} = 2(\sec\theta\tan\theta - \text{cosec}\theta\cot\theta) = 0$$ **[1]**

$$\frac{\sin\theta}{\cos^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

$$\sin^3\theta = \cos^3\theta$$

$$\tan^3\theta = 1$$

$$\theta = \frac{\pi}{4}$$ **[1]**

$$\frac{d^2z}{d\theta^2} = 2\left[\sec^3\theta + \sec\theta\tan^2\theta + \text{cosec}^3\theta + \text{cosec}\theta\cot^2\theta\right]$$

$$\frac{d^2z}{d\theta^2} > 0 \text{ at } \theta = \frac{\pi}{4}$$

So Z is minimum **[1]**

Minimum value of $Z = 2\left(\sec\frac{\pi}{4} + \text{cosec}\frac{\pi}{4}\right)$

$$Z = 4\sqrt{2}$$ **[1]**

27. Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ on $\left[0, \frac{\pi}{2}\right]$. **[CBSE 2016, 6M]**

OR

Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Sol. $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$

$$y' = \frac{4(2+\cos\theta)\cos\theta - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{4[2\cos\theta + \cos^2\theta + \sin^2\theta]}{(2+\cos\theta)^2} - 1$$

$$= \frac{4(1+2\cos\theta)}{(2+\cos\theta)^2} - 1 = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \quad ...(1)$$ [2]

$\because \cos\theta \geq 0$ if $\theta \in \left[0, \frac{\pi}{2}\right]$

$4 - \cos\theta > 0$ if $\theta \in \left[0, \frac{\pi}{2}\right]$

$(2+\cos\theta)^2 > 0$ if $\theta \in \left[0, \frac{\pi}{2}\right]$

$\therefore \quad y' \geq 0$ in $\left[0, \frac{\pi}{2}\right]$

$\Rightarrow \quad$ y is increasing function. ...(2) [2]

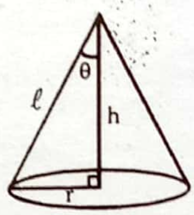
OR

Let radius of cone = r

height of cone = h

slant height of cone = ℓ

and semi-vertical angle = α [1]



Volume of cone $v = \frac{1}{3}\pi r^2 h = \frac{\pi\ell^3}{3}\sin^2\theta\cos\theta$ [1]

$$\frac{dv}{d\theta} = \frac{\pi\ell^3}{3}\left[2\sin\theta\cos^2\theta - \sin^3\theta\right]$$ [1]

for maxima and minima $\frac{dv}{d\theta} = 0$ [½]

$$\frac{dv}{d\theta} = \frac{\pi\ell^3}{3}\left[2\sin\theta\cos^2\theta - \sin^3\theta\right] = 0$$

$\therefore \quad 2\cos^2\theta - (1 - \cos^2\theta) = 0$

$\cos\theta = \frac{1}{\sqrt{3}}$

$\theta = \cos^{-1}\frac{1}{\sqrt{3}}$ [1½]

$$\frac{d^2v}{d\theta^2} = \frac{\pi\ell^3}{3}\left[2\cos^3\theta - 7\sin^2\theta\cos\theta\right]$$

$\frac{d^2v}{d\theta^2}$ is negative at $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$

hence v is maximum at $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$ [1]

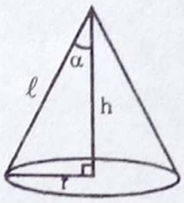
Aliter :

Let radius of cone = r

height of cone = h [1]

slant height of cone = ℓ

and semi-vertical angle = α



Volume of cone $v = \frac{1}{3}\pi r^2 h$ ..... (1)

$\because \quad \ell^2 = r^2 + h^2$

$\therefore \quad r^2 = \ell^2 - h^2$ ..... (2)

Putting the value of r^2 in equation (1)

$$v = \frac{1}{3}\pi(\ell^2 - h^2).h$$

$$v = \frac{1}{3}\pi(\ell^2 h - h^3) \quad \text{..... (3)}$$ [1]

$$\therefore \quad \frac{dv}{dh} = \frac{1}{3}\pi(\ell^2 - 3h^2) \quad \text{..... (4)}$$ [1]

For stationary point, $\frac{dv}{dh} = 0$ [½]

$\frac{1}{3}\pi(\ell^2 - 3h^2) = 0 \Rightarrow \quad h = \frac{\ell}{\sqrt{3}}$

Now $\frac{d^2v}{dh^2} = \frac{1}{3}\pi(-6h) = -2\pi h$

$$\therefore \quad \left(\frac{d^2v}{dh^2}\right)_{\text{at } h=\frac{\ell}{\sqrt{3}}} = -2\pi.\frac{\ell}{\sqrt{3}} < 0$$

$\therefore \quad$ Volume is maximum at $h = \frac{\ell}{\sqrt{3}}$ [1]

Putting h in equation (2), we get $r^2 = \ell^2 - \frac{\ell^2}{3}$

$$r^2 = \frac{2\ell^2}{3} \quad \Rightarrow \quad r = \frac{\sqrt{2}\,\ell}{\sqrt{3}}$$ [1]

Now, $\tan\alpha = \frac{r}{h}$

$$\tan\alpha = \frac{\sqrt{2}\ell/\sqrt{3}}{\ell/\sqrt{3}} = \sqrt{2}$$

$\because \quad \alpha = \tan^{-1}\sqrt{2}$

$\therefore \quad \alpha = \cos^{-1}\frac{1}{\sqrt{3}}$

$y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

$y = 2\log\left(\frac{x+1}{\sqrt{x}}\right)$

$y = 2\log(x+1) - 2\log\sqrt{x}$

$\frac{dy}{dx} = \frac{2}{x+1} - \frac{2}{\sqrt{x}} \times \frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = \frac{2}{x+1} - \frac{1}{x}$

$\frac{dy}{dx} = \frac{2x-(x+1)}{x(x+1)} = \frac{x-1}{x(x+1)}$

$\frac{dy}{dx} = \frac{x-1}{x(x+1)}$

$\frac{d^2y}{dx^2} = \frac{x(x+1)(1)-(x-1)(2x+1)}{x^2(x+1)}$

$\frac{d^2y}{dx^2} = \frac{x(x+1)}{x^2(x+1)} - \frac{(x-1)(2x+1)}{x^2(x+1)}$

$= \frac{1}{x} - \frac{(x-1)(2x+1)}{x^2(x+1)}$

$x(x+1)^2 \times \left(\frac{x(x+1)-(x-1)(2x+1)}{x^2(x+1)}\right) + (x+1)^2$

$\frac{(x+1)}{x}\left(x^2+x-(2x^2+x-2x-1)\right) + x$

0	30	45	60	90
0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1