Discrete Structure.

Unit-2 Propositional Logic

Introduction to Propositional Logic :-

Propositional logic is logic at the sentential level. The smallest unit we deal with in propositional logic is a sentence. We do not go inside individual sentences and analyze or discuss their meanings. We are going to be interested only in true or false of statements and major concern is whether or not the truth or false of a certain sentence. Thus sentences considered in this logic are not arbitrary sentences but are the ones that are true or false. These kind of sentences are called prepositions.

"A proposition is a declarative statement that is sufficiently objective, meaningful and precise to have a truth value. [True or False] So, a proposition is a declarative sentence that is either true or false but not both."

These two values true and false are denoted by 'T' and 'F' respectively or sometimes 0 and 1.

Exp.1 Which of the following sentences are statement?

a) Chandigarh is the capital of Panjab.
b) $2+2=5$
c) Where are you going?
d) Bring a cup of coffee for me?
e) The square of 3 is 9.
f) $2<5$.
g) What a beautiful place!
h) Come here;
i) $x^2=26$

Solⁿ. a, b, e, f are prepositions.
a, is true, b is false, e is true & f is also true
c is a question, so it is not a statement
d, h is command, so not a statement
g is an exclamatory sentence, it is not having any truth value.
i is a assignment, so it is also not a statement.

A proposition that is true, under all circumstance is refferred to as a <u>Tautology</u> & a proposition that is false under all circumstances is refferred to as a <u>Contradiction</u> or <u>absurdity</u> and if a preposition that is neither a tautology nor a contradiction is called a <u>Contingency</u>.

Propositional Variable

The Lowercase letter p, q, r --- used to denote the propositions are called propositional variable that is, variables that can be replaced by propositions.

Compound Propositions

Statements or propositional variables can be combined by logical connectives to obtain compound statements. A preposition consisting of only a single propositional variable or a single propositional constant is called an <u>atomic</u> or primitive proposition, that is they can not be further subdivided.

Connectives.

Connectives are those notations which are used to get new propositions from the given propositions.
Let us consider the following propositions.

① **Negation** (NOT) denoted by $\neg$ or $\sim$

If a preposition is denoted by P, then its negation is denoted by $\neg P$ or $\sim P$.

P	$\neg P$
T	F
F	T

② **Conjunction** (AND) denoted by ($\wedge$)

If P and Q are two prepositions then conjunction (AND) of P and Q is also a preposition.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

③ **Disjunction** (OR) denoted by ($\vee$)

If P and Q are two prepositions then disjunction (OR) of P and Q is also a preposition.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

4) Implication (IF--Then) or conditional $\Rightarrow$ or $\rightarrow$
Let P and Q be two propositions, we define a preposition, "IF P Then Q" denoted by $P \Rightarrow Q$, which is true if both P and Q are true or if P is false.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

5) If and only if denoted by $\Leftrightarrow$
If P and Q are two given propositions, then we define a preposition, "P if and only if Q" denoted by $P \Leftrightarrow Q$ which is true if both P and Q are true or if both P and Q are false.

P	Q	$P \Leftrightarrow Q$
F	F	T
F	T	F
T	F	F
T	T	T

6) NOR It is negation of OR denoted by $\downarrow$ so for any two Preposition P and Q

$$P \downarrow Q = \neg (P \vee Q)$$

7) NAND - Similarly NAND stands for negation of AND. The corrective NAND is denoted by $\uparrow$

$$\boxed{P \uparrow Q = \neg (P \wedge Q)}$$

Exp 2. Let p, q, r denotedes the following simple statements

p: Ram goes out for a walk.

q: The moon is out

r: It is snowing

Then write the following coumpound statements in symbolic form.

① If the moon is out and it is not snowing, then Ram goes out for a walk.

② If the moon is out, then if it is not snowing, Ram goes out for a walk.

③ It is not the case that Ram goes out for a walk if and only if it is snowing or the moon is out.

④ Ram will go out walking if and only if the moon is out.

⑤ If it is snowing and the moon is not out, then Ram will not go out for a walk.

Solⁿ (i) $(q \wedge \neg r) \Rightarrow p$ ② $q \Rightarrow (\neg r \Rightarrow p)$

③ $\neg(p \Leftrightarrow r \vee q)$ ④ $p \Leftrightarrow q$

⑤ $(r \wedge \neg q) \Rightarrow \neg p$

Q3 Assign a truth value to each of the following

(i) $0<0 \vee 15<8$ (ii) $5\times 4=20 \vee 0>2$ (iii) $6+4=10 \wedge 0<2$

Solⁿ. (i) False, since both false

(ii) True, since one of its is true

(iii) True, both true.

Q.4 If p: It is cold and q: It is raining

Write the verbal sentence which describe each of the following statement:

a) $\neg p$ (b) $p \wedge q$ (c) $p \vee q$ (d) $p \vee \neg q$

a) $\neg p$: It is not cold (b) It is cold and raining

c) It is cold or raining (c) It is cold or it is not raining.

Q5. Express the following statement in symbolic form. "If p implies q and q implies r then p implies r"

Sol^n. $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$

For this question first do Q.8 then 7 then 6.

Q6. Let p be the proposition "The earth is flat". Let q be "All birds sing" and let r be "Manhattan is an island". Write out the following proposition.

Sol^n (a) $\sim q \wedge r$ (b) $\sim(q \wedge p)$ (c) $r \vee (p \wedge q)$
(d) $p \wedge \sim(q \vee r)$

Sol^n. (a) $\sim q \wedge r$: Some birds do not sing and Manhattan is an island.

(b) $\sim(q \wedge p)$: It is not the case that all birds sing and earth is flat.

(c) $r \vee (p \wedge q)$: Either Manhattan is an island or it is the case both earth is flat and all birds sing.

(d) $p \wedge \sim(q \vee r)$: The earth is flat and it is not the case, that either all birds sing or that Mahhattan is an island.

Q.7 Find the negation of proposition p & q.
p: All people are intelligent
q: No student is graduate.

Sol^n. $\sim p$: Some people are not intelligent
OR $\sim p$: There exist a people who is not intelligent
OR $\sim p$: There exist people who are not intelligent
or $\sim p$: At least one people is not intelligent

Q8 Write the negate of "No dogs have three legs."

Sol^n. A good way to start when negating any preposition P is to assert ¬P, i.e If we have a sentence "P", we can negate it simply by writing, "It is not the case that P".

P: "No dogs have three legs".

then ¬P: It is not the case that No dogs have 3 legs

or There exist ~~some~~ one or more dogs with three legs".

or: Some dogs have three legs.

Q: Some animals do not eat meat.

¬Q: It is not the case that some animals do not eat meat.

OR: "All animals eat meat" Ans.

Proposition and Truth Table

A truth table shows the truth value of a compou proposition for all possible cases.

Q9 Construct a truth table for each compound proposi

(i) $(p \wedge q) \vee (p \wedge r)$ (ii) $\sim(p \vee q) \vee (\neg p \wedge \neg q)$

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

(ii) $\neg(p \vee q) \vee (\neg p \wedge \neg q)$

p	q	r	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p \wedge \neg q)$	Result
0	0	0	0	1	1	1	1	1
0	0	1	0	1	1	1	1	1
0	1	0	1	0	1	0	0	0
0	1	1	1	0	1	0	0	0
1	0	0	1	0	0	1	0	0
1	0	1	1	0	0	1	0	0
1	1	0	1	0	0	0	0	0
1	1	1	1	0	0	0	0	0

$\neg(p\vee q)\vee(\neg p\wedge\neg q)$

(ii)

p	q	$\neg p$	$\neg q$	$(p\vee q)$	$\neg(p\vee q)$	$(\neg p\wedge\neg q)$	$\neg(p\vee q)\vee(\neg p\wedge\neg q)$
0	0	1	1	0	1	1	1
0	1	1	0	1	0	0	0
1	0	0	1	1	0	0	0
1	1	0	0	1	0	0	0

Ans.

Q.10 Write out the truth table for the following proposition.

(a) $P\vee(Q\wedge\neg R)$

(b) $(\neg P\vee Q)\vee\neg R$

(c) $(\neg P\wedge Q)\vee(R\wedge T)$

Q.11 For each of the following pairs of propositional forms find an assignment of truth values that shows the member not to be equivalent:

(a) $\neg P\wedge Q$ and $\neg(P\wedge Q)$

(b) $(P\wedge Q)\vee R$ and $P\wedge(Q\vee R)$

(c) $\neg P\wedge\neg Q$ and $\neg(P\wedge Q)$

Sol^n.

P	Q	$\neg P$	$\neg P\wedge Q$	$P\wedge Q$	$\neg(P\wedge Q)$
0	0	1	0 ✓	0	1 ✓
0	1	1	1	0	1
1	0	0	0 ✓	0	1 ✓
1	1	0	0	1	0

when $P=0$ and $Q=0$ & $P=1$ and $Q=0$, they are not equivalent.

(iii)

P	Q	$\neg P$	$\neg Q$	$\neg P\wedge\neg Q$	$(P\wedge Q)$	$\neg(P\wedge Q)$
0	0	1	1	1 ✓	0	1 ✓
0	1	1	0	0	0	1
1	0	0	1	0	0	1
1	1	0	0	0 ✓	1	0 ✓

when $P=0$ and $Q=1$ & $P=1$ and $Q=0$ they are not equivalent.

Q12 Show that $P \Leftrightarrow Q$ and $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ are equivalent.

P	Q	$P \Leftrightarrow Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
0	0	1	1	1	1
0	1	0	1	0	0
1	0	0	0	1	0
1	1	1	1	1	1

Hence both are equivalent.

Q.13 Verify that the two proposition $((P \wedge q) \vee (P \wedge r)) \vee s$ and $(\neg p \vee (\neg q \wedge \neg r)) \vee s$ are equivalent

Ans.

P	Q	r	s	$(P \wedge q)$	$(P \wedge r)$	$(P \wedge q) \vee (P \wedge r)$	$(P \wedge q) \vee (P \wedge r) \Rightarrow s$
0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	1
0	0	1	0	0	0	0	1
0	0	1	1	0	0	0	1
0	1	0	0	0	0	0	1
0	1	0	1	0	0	0	1
0	1	1	0	0	0	0	1
0	1	1	1	0	0	0	1
1	0	0	0	0	0	0	1
1	0	0	1	0	0	0	1
1	0	1	0	0	1	1	0
1	0	1	1	0	1	1	1
1	1	0	0	1	0	1	0
1	1	0	1	1	0	1	1
1	1	1	0	1	1	1	0
1	1	1	1	1	1	1	1

$\neg p$	$\neg q$	$\neg r$	$\neg q \wedge \neg r$	$\neg p \vee (\neg q \wedge \neg r)$	$\neg p \vee (\neg q \wedge \neg r) \vee s$
1	1	1	1	1	1
1	1	1	1	1	1
1	1	0	0	1	1
1	1	0	0	1	1
1	0	1	0	1	1
1	0	1	0	1	1
1	0	0	0	1	1
1	0	0	0	1	1
0	1	1	1	1	1
0	1	1	1	1	1
0	1	0	0	0	0
0	1	0	0	0	1
0	0	1	0	0	0
0	0	1	0	0	1
0	0	0	0	0	0
0	0	0	0	0	1

Q14 Verify that the proposition $p \vee \neg(p \wedge q)$ is tautology

Ans.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
0	0	0	1	1
0	1	0	1	1
1	0	0	1	1
1	1	1	0	1

all 1's.

Q15 Determine whether each of the following is a tautology contradiction or a contigency.

(a) $(P \vee \neg Q) \Rightarrow (P \wedge Q)$

(b) $(\neg P \wedge (P \vee Q)) \Rightarrow Q$

(c) $(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

(a) $(P \vee \neg Q) \Rightarrow (P \wedge Q)$

P	Q	$\neg Q$	$P \vee \neg Q$	$P \wedge Q$	$(P \vee \neg Q) \Rightarrow (P \wedge Q)$
0	0	1	1	0	0
0	1	0	0	0	1
1	0	1	1	0	0
1	1	0	1	1	1

This is Contigency.

(b) $(\neg P \wedge (P \vee Q)) \Rightarrow Q$

P	Q	$\neg P$	$P \vee Q$	$(\neg P \wedge (P \vee Q))$	$(\neg P \wedge (P \vee Q)) \Rightarrow Q$
0	0	1	0	0	1
0	1	1	1	1	1
1	0	0	1	0	1
1	1	0	1	0	1

Tautology

(c) $(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

P	Q	$P \Rightarrow Q$	$\neg Q$	$P \wedge \neg Q$	$(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	1	0
1	1	1	0	0	0

Contradiction OR absurdity.

Converse, Contrapositive and Inverse

If P and Q are proposition then, the

Converse of $P \Rightarrow Q$ is the implication $Q \Rightarrow P$

Contrapositive of $P \Rightarrow Q$ is the implication $\neg Q \Rightarrow \neg P$

Inverse of $P \Rightarrow Q$ is the implication $\neg P \Rightarrow \neg Q$

Q16 Consider the following conditional statement P: If the floods destroy my house or the fire destroy my house, then my insurance company will pay me. Write converse, inverse and contrapositive of the statement

Soln. Let r: Floods destroy my house
s: Fires destroy my house
t: my insurance company will pay me.

In symbolic form, the given statement canbe written as

$$P: (r \vee s) \Rightarrow t$$

(i) Converse is $t \Rightarrow (r \vee s)$

If my insurance comp. pay me, then either Flood destroy my house or fires destroy my house.

(ii) Inverse $\neg(r \vee s) \Rightarrow \neg t$

$$\neg r \wedge \neg s \Rightarrow \neg t$$

$\therefore$ If Floods and Fires do not destroy my house then, my insurance company will not pay me.

(iii) Contrapositive $\neg t \Rightarrow \neg(r \vee s)$

$$\neg t \Rightarrow \neg r \wedge \neg s$$

If my insurance company does not pay me then floods and fires does not destroy my house.

Q17 Show that Contrapositives are logically equivalent

i.e $\sim Q \Rightarrow \neg P \equiv P \Rightarrow Q$

P	Q	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$	$P \Rightarrow Q$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	1	1

Q18 Write Contrapositive, converse and inverse of the following.

"Indian team wins whenver match is played in Kolkata home town of Ganguly".

Sol^n.

Let P: Indian team wins

Q: Match is played in Kolkata, home town of Ganguly

Then the given statement is P whenever Q.

$\Rightarrow \quad Q \Rightarrow P$

∴ "If match is in Kolkata, home town of Ganguly, the Indian team will win"

(i) The Converse of $Q \Rightarrow P$ is $P \Rightarrow Q$.

"If indian team wins then match is in Kolkata, home town of Ganguly"

(ii) Inverse of $Q \Rightarrow P$ is $\neg Q \Rightarrow \neg P$

"If the match is not played in Kolkata, home town of Ganguly the Indian team ~~wi~~ does not win."

(iii) Contrapositive:- of $Q \Rightarrow P$ is $\neg P \Rightarrow \neg Q$

"If Indian team does not wins then match is not played in Kolkata, home town of Ganguly"

Algebra of Propositions

Idempotent Law 1. (a) $P \vee P = P$

(b) $Q \wedge Q = Q$

Commu~~nica~~tative Law 2. (a) $P \vee Q \equiv Q \vee P$

(b) $p \wedge q \equiv q \wedge p$

Associative Law 3. (a) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

(b) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive Law 4 (a) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Identity Law 5 (a) $p \vee T = \text{True}$

(b) $p \vee \text{False} = P$

(c) $P \wedge T = P$

(d) $P \wedge \text{False} = F$

Complement Law 6 (a) $P \vee \neg P = \text{True}$

(b) $P \wedge \neg P = \text{False}$

Involution Law 7. (a) $\neg\neg P = P$

Demorgan's Law 8. (a) $\neg(P \vee Q) = \neg P \wedge \neg Q$

(b) $\neg(P \wedge Q) = \neg P \vee \neg Q$

Absorption Law 9. (a) $P \vee (P \wedge Q) = P$

(b) $P \wedge (P \vee Q) = P$

Contrapositive Law 10. (a) $P \Rightarrow Q = \neg Q \Rightarrow \neg P$

11 (a) $P \Rightarrow Q \equiv (\neg P \vee Q)$

12. $(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q) \Rightarrow P$

Q19 Use the Laws to show that $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$

Solⁿ: $\neg(p \vee q) \vee (\neg p \wedge q)$

$= (\neg p \wedge \neg q) \vee (\neg p \wedge q)$

$= \neg p \wedge (\neg q \vee q)$

$= \neg p \wedge T$

$= \neg p$ Ans

Q20 Prove that $(P \Rightarrow q) \wedge (R \Rightarrow q) \equiv (P \vee R) \Rightarrow q$

$= (P \Rightarrow q) \wedge (R \Rightarrow q)$

$= (\neg P \vee q) \wedge (\neg R \vee q)$

$= (\neg P \wedge \neg R) \vee q$

$= \neg(P \vee R) \vee q$

$= (P \vee R) \Rightarrow q$ By Rule (11)

Q.21 Show that $\{[(p \vee q) \Rightarrow r] \wedge (\neg q)\} \Rightarrow (q \Rightarrow r)$ is a tautology without using truth Table

Solⁿ:

$\{[(p \vee q) \Rightarrow r] \wedge (\neg q)\} \Rightarrow (q \Rightarrow r)$

$= \{[(\neg(p \vee q)) \vee r] \wedge (\neg q)\} \Rightarrow (\neg q \vee r)$

$= \{[(\neg p \wedge \neg q) \vee r] \wedge \neg q\} \Rightarrow (\neg q \vee r)$

$= \{(\neg p \wedge \neg q \wedge \neg q) \vee (r \wedge \neg q)\} \Rightarrow (\neg q \vee r)$

$= (\neg p \wedge \neg q) \vee (r \wedge \neg o$

$$\{[(p \vee q) \Rightarrow r] \wedge (\neg p)\} \Rightarrow (q \Rightarrow r)$$

$= \{[\neg(p \vee q) \vee r] \wedge (\neg p)\} \Rightarrow (\neg q \vee r)$

$= \{[(\neg p \wedge \neg q) \vee r] \wedge (\neg p)\} \Rightarrow (\neg q \vee r)$ Distributive Law

$= \{[\neg p \wedge \neg q \wedge \neg p] \vee (r \wedge \neg p)\} \Rightarrow (\neg q \vee r)$ Asso.

$= (\neg p \wedge \neg q) \vee (r \wedge \neg p) \Rightarrow (\neg q \vee r)$

$= \neg p \wedge (\neg q \vee r) \Rightarrow (\neg q \vee r)$

$= \neg(\neg p \wedge (\neg q \vee r)) \vee (\neg q \vee r)$

$= [p \vee \neg(\neg q \vee r)] \vee (\neg q \vee r)$

$= p \vee [\neg(\neg q \vee r) \vee (\neg q \vee r)]$ associative Law

$= \cancel{p \vee (\neg q \vee r)}$

$= p \vee T$

$= T$ Hence Prove.

Q22 Is the statement tautology: $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$

$$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$$

$\Rightarrow [(\neg p \vee q) \wedge (\neg q \vee r)] \Rightarrow (\neg p \vee r)$

$\Rightarrow \neg[(\neg p \vee q) \wedge (\neg q \vee r)] \vee (\neg p \vee r)$

$\Rightarrow [\neg(\neg p \vee q) \vee \neg(\neg q \vee r)] \vee (\neg p \vee r)$

$\Rightarrow [(p \wedge \neg q) \vee (q \wedge \neg r)] \vee (\neg p \vee r)$

Q.23

Simplify the left side of each of the following to obtain R·H·S

(a) $[(P\wedge Q)\vee(P\wedge\neg Q)\vee(\neg P\wedge Q)] \Leftrightarrow P\vee Q$

$$\begin{aligned}
L\cdot H\cdot S &= [(P\wedge Q)\vee(P\wedge\neg Q)\vee(\neg P\wedge Q)]\\
&= [P\wedge\underline{(Q\vee\neg Q)}\vee(\neg P\wedge Q)]\\
&= [P\vee(\neg P\wedge Q)]\\
&= \underbrace{(P\vee\neg P)}_{T}\wedge(P\vee Q)\\
&= (P\vee Q)
\end{aligned}$$

(b) $[(P\wedge Q\wedge R)\vee(P\wedge\neg Q\wedge R)\vee(P\wedge Q\wedge\neg R)] \Leftrightarrow P\wedge(\neg Q\Rightarrow R)$

$$\begin{aligned}
L\cdot H\cdot S &= [(P\wedge Q\wedge R)\vee(P\wedge\neg Q\wedge R)\vee(P\wedge Q\wedge\neg R)]\\
&= \{\{P\wedge\underbrace{[(Q\wedge R)\vee(\neg Q\wedge R)]}_{t}\}\vee[P\wedge\underbrace{(Q\wedge\neg R)}_{q}]\}\\
&= P\wedge[\cancel{Q\wedge(R\vee}(Q\wedge R)\vee(\neg Q\wedge R)\vee(Q\wedge\neg R)]\\
&= P\wedge[(Q\wedge R)\vee(Q\wedge\neg R)\vee(\neg Q\wedge R)]\\
&= P\wedge[Q\wedge(R\vee\neg R)\vee(\neg Q\wedge R)]\\
&= P\wedge[(Q\wedge T)\vee(\neg Q\wedge R)]\\
&= P\wedge[Q\vee(\neg Q\wedge R)]\\
&= P\wedge[(Q\vee\neg Q)\wedge(Q\vee R)]\\
&= P\wedge[Q\vee R]\\
&= P\wedge[\neg(\neg Q)\vee R)]\\
&= P\wedge[\neg Q\Rightarrow R] = R\cdot H\cdot S
\end{aligned}$$

Q.24 (a) Write $\neg P$, $P\wedge q$, and $P\vee q$ in terms of nor operator alone.

(b) write $p\vee(q\wedge r)$ in terms of the nor operator alone

Solⁿ (a) $\neg P \equiv \neg(P\vee P)$ $\qquad\because \neg(P\vee q) = P\downarrow q$

$\equiv P\downarrow P$

$(P\wedge q) \equiv \neg(\neg P\vee\neg q) \equiv \neg P\downarrow\neg q \equiv (P\downarrow P)\downarrow(q\downarrow q)$

$(P\vee q) \equiv \neg(\neg P\wedge\neg q) \equiv \neg(t\wedge u) \equiv \neg[(P\downarrow P)\downarrow(q\downarrow q)]$

$(p \vee q) = \neg[\neg(p \vee q)] = \neg(p \downarrow q) \equiv (p \downarrow q) \downarrow (p \downarrow q)$ $\quad \because \neg p = p \downarrow p$

(b) $p \vee (q \wedge r) \equiv \neg[\neg(p \vee (q \wedge r))]$

$= \neg[\neg(p \vee (q \wedge r))]$

$\equiv \neg\{\neg(p \vee [(q \downarrow q) \downarrow (r \downarrow r)])\}$

$= p \vee [(q \downarrow q) \downarrow (r \downarrow r)]$

$\equiv (p \downarrow [(q \downarrow q) \downarrow (r \downarrow r)]) \downarrow (p \downarrow [(q \downarrow q) \downarrow (r \downarrow r)])$ Ans

Q25 The Nand operator $p \uparrow q$ is defined as

$p \uparrow q \equiv \neg(p \wedge q)$

(a) Show that $p \uparrow p$ is equivalent to $\neg p$

(b) Write $p \wedge q$ in terms of the nand operator alone.

(c) Write $p \vee q$ in terms of the nand operator alone.

Solⁿ.

(a) $p \uparrow p \equiv \neg(p \wedge p) \equiv \neg(p) \equiv \neg p$

(b) $(p \wedge q) \equiv \cancel{\neg(\neg p \vee \neg q)} = \neg(\neg(p \wedge q))$

$= \neg(p \uparrow q)$ $\cancel{\equiv (\neg p) \uparrow (\neg q)}$

$\cancel{= (p \uparrow p) \uparrow q \uparrow q}$

$= \neg \underbrace{(p \uparrow q)}_{\text{let } t} \equiv \neg t = t \uparrow t$

= put the value of t

$= (p \uparrow q) \uparrow (p \uparrow q)$

(c) $(p \vee q) \equiv \neg(\neg p \wedge \neg q) \equiv \neg[(p \uparrow p) \wedge (q \uparrow q)]$

$\equiv (p \uparrow p) \uparrow (q \uparrow q)$ Ans $\because \neg(p \wedge q) \equiv p \uparrow q$

Normal Forms

By comparing truth tables, we can determine whether two logical expressions P and Q are equivalent or not, but the process is very difficult when the number of variables increases. A better option is to transform the expression P and Q to standard form of expression P' and Q' and a simple comparison of P' and Q' show that whether $P \equiv Q$. These standard form are called canonical forms or normal forms.

use product for conjunction &
sum for disjunction.

Four types of Normal form.

1. Disjunctive Normal Form (DNF)
2. Conjunctive Normal Form (CNF)
3. Principal Disjunctive Normal Form (PDNF)
4. Principal Conjuctive Normal form. (PCNF)

① Disjunctive Normal Form

A logical expression is said to be in disjunctive normal form if it a sum of elementary products.
For exp. $p \vee (q \wedge r)$, $p \vee (\neg q \wedge r)$.

But $p \wedge (q \vee r)$ is not a disjunctive normal form.

For exp. $P \oplus Q$

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

$p \oplus q = pq' + p'q$

it is ~~product~~ sum of Product

it is in disjuctive N.F.

In general, we obtain the disjuctive normal form for an n-variable propositional form $f(P_1, P_2, P_3, --- P_n)$ from its truth table as follows:

For each row in which $f(P_1, P_2, P_3, -- P_n)$ assume the value 1, we form the conjuction $P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n$ where we take P_K if there is a 1 in the k-th position in the row and $\neg P_K$ if there is a 0 there.

eg. Find the disjunctive normal form for the propositional form $f(P, Q, R)$ defined as.

P	Q	R	f(P,Q,R)
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

DNF $\equiv$

$$(\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R) \wedge (P \wedge Q \wedge R)$$

Note:- Construction a disjuctive normal form of a given formula.

1. Eliminate $\rightarrow$ and $\Leftrightarrow$ by using Logical connectives
2. Use Demorgen's Law to eliminate $\neg$ before sum or products
3. Apply distributive Law repeatedly to eleminate product to sum.

Q. obtain DNF of the formula $P \wedge (P \Rightarrow Q)$

$P \wedge (\neg P \vee Q)$ eliminate $\Rightarrow$

$(P \wedge \neg P) \vee (P \wedge Q)$ apply distributive Law

Q. obtain disjunctive normal form

$$p \vee (\neg p \Rightarrow (q \vee (q \Rightarrow \neg r)))$$

$\Rightarrow p \vee (\neg p \Rightarrow (q \vee (\neg q \vee \neg r)))$

$\Rightarrow p \vee (p \vee (q \vee (\neg q \vee \neg r)))$

$\Rightarrow p \vee p \vee q \vee (\neg q \vee \neg r)$

$\Rightarrow p \vee q \vee \neg q \vee \neg r$ Ans

② Conjunctive Normal Form.

A expression is said to be in conjunctive normal form if it is a product of sums

For: exp $p \wedge r,\ p \wedge (q \vee r)$ ...

Q. obtain C.N.F

(a) $p \wedge (p \Rightarrow q)$ (b) $(q \vee (p \wedge r)) \wedge \neg((p \vee r) \wedge q)$

(a) $p \wedge (p \Rightarrow q) \equiv p \wedge (\neg p \vee q)$

$\equiv$ Ans

(b) $(q \vee (p \wedge r)) \wedge \neg((p \vee r) \wedge q)$

$\Rightarrow q \vee (p \wedge r) \wedge \neg(p \vee r) \vee \neg q$

$\Rightarrow (q \vee p) \wedge (q \vee r) \wedge (\neg p \wedge \neg r) \vee \neg q$

$\Rightarrow (q \vee p) \wedge (q \vee r) \wedge (\neg p \vee \neg q) \wedge (\neg r \vee \neg q)$

③ Principal disjunctive Normal form :- or Sum of Product Canonical form.

The advantages of constructing principal disjunctive normal form are:

1. For a given formula, its principal disjunctive normal form is unique.
2. Two formulas are equivalent if and only if their principal disjunctive normal forms concides.

A Logical expression is said to be in principal disjunctive normal form if it is a sum of minterms.

We have two method to obtain principal disjunctive normal form :-

① Truth Table [take mintrum where function is equal to 1]

② Without Truth table

(i) obtain disjunctive normal form

(ii) Drop elementary products which are contradiction such as $(P \wedge \neg P)$.

(iii) If P_i and $\neg P_i$ are missing in an elementry product α, replace α by $(\alpha \wedge P_i) \vee (\alpha \wedge \neg P_i)$.

(iv) Repeat Step (iii), untill all elementary products are reduced to sum of minterms. Use idempotent Law to avoide repetition of minterm

Q. Obtain the Principal disjunctive NF of formula $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

Solⁿ: ① There is no need to apply step 1 because it is already in disjunctive normal form.

② There is no need to apply step ② since there in no elementary product.

$P \wedge Q$ is replaced by $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R)$
$(\neg P \wedge R)$ " " " $(\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q)$
$(Q \wedge R)$ " " " $(Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$

Thus the formula is replaced by

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q) \vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$$

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg R \wedge R \wedge \neg Q)$$

(4) Principal Conjunctive Normal Form (PCNF)

→ Product of sum canonical form.

For finding principal C.N.F we have to first find PDNF of $\neg\alpha$ and then apply negation.

Q. Find the principal CNF ~~and~~ of $p \vee (q \Rightarrow r)$

Solⁿ

Suppose $\alpha = p \vee (q \Rightarrow r)$

$\Rightarrow p \vee (\neg q \vee r)$

$(\neg\alpha) \Rightarrow \neg(p \vee (\neg q \vee r))$

$\Rightarrow \neg p \wedge \neg(\neg q \vee r)$

$\Rightarrow \neg p \wedge q \wedge \neg r$

this is the PDNF of $(\neg\alpha)$

$\therefore$ PCNF of (α) is $\neg(\neg p \wedge q \wedge \neg r)$

$= p \vee \neg Q \vee \neg r$

Ans

PCNF $\Rightarrow$ Product of sum (max term) make zero.
PDNF $\Rightarrow$ Sum of Product (minterm) make one.

Q. obtain the principal conjunctive normal form
(a) $p \wedge q$ using truth table and without truth table.
(c) $(\neg p \Rightarrow r) \wedge (q \Leftrightarrow p)$

(a) $p \wedge q$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Take maxterm (zero)
Required PCNF = $(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q)$

(b) $(\neg p \Rightarrow r) \wedge (q \Leftrightarrow p)$

$\Rightarrow (p \vee r) \wedge [(q \Rightarrow p) \wedge (p \Rightarrow q)]$

$\Rightarrow (p \vee r) \wedge [(\neg q \vee p) \wedge (\neg p \vee q)]$

$\Rightarrow$ This is in (~~sum of Product form~~) Product of Sum ~~Disjunctive~~ Conjunctive Normal form.

$\Rightarrow [(p \vee r) \vee (q \wedge \neg q)] \wedge [(\neg q \vee p) \vee (r \wedge \neg r)] \wedge [(\neg p \vee q) \vee (r \wedge \neg r)]$

$\Rightarrow [(p \vee r \vee q) \wedge (p \vee r \vee \neg q)] \wedge [(\neg q \vee p \vee r) \wedge (\neg q \vee p \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)]$

$\Rightarrow (p \vee r \vee q) \wedge (p \vee r \vee \neg q) \wedge (\neg q \vee p \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$

p	q	r	$\neg p$	$\neg p \Rightarrow r$	$q \Leftrightarrow p$	$\wedge$
0	0	0	1	0	1	0
0	0	1	1	1	1	1
0	1	0	1	0	0	0
0	1	1	1	1	0	0
1	0	0	0	1	0	0
1	0	1	0	1	0	0
1	1	0	0	1	1	1
1	1	1	0	1	1	1

Maxterm (PCNF) = $(p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$

Arguments

Argument is a sequence of statements. All the statements except the final one are called premises and final statement is called conclusion.

An argument is said to be logically valid if and only if the conjunction of the premises implies the conclusion, i.e all premises are true.

i.e $P_1 \wedge P_2 \wedge P_3 \cdots \wedge P_n \Rightarrow C$ is a tautology.

It can be expressed as two forms.

① $P_1, P_2, P_3 \cdots P_n \vdash C$ (Tautology form)

② $P_1 \wedge P_2 \wedge P_3 \cdots \wedge P_n \Rightarrow C$ (inferential form).

$$\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline C \\ \hline \end{array}$$

An argument which is not valid is called fallacy.

Note: To check the validity of a given argument is to show that the statement $P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n \Rightarrow C$ is a tautology.

Q. Test the validity of the following argument:

$$p \Rightarrow q,\ r \Rightarrow \neg q \vdash (p \Rightarrow \neg r)$$

p	q	r	$\neg q$	$\neg r$	$P_1: p\Rightarrow q$	$P_2: r\Rightarrow \neg q$	$P_1 \wedge P_2$	$c: p\Rightarrow \neg r$	$P_1 \wedge P_2 \Rightarrow c$
0	0	0	1	1	1	1	1	1	1
0	0	1	1	0	1	1	1	1	1
0	1	0	0	1	1	1	1	1	1
0	1	1	0	0	1	0	0	1	1
1	0	0	1	1	0	1	0	1	1
1	0	1	1	0	0	1	0	0	1
1	1	0	0	1	1	1	1	1	1
1	1	1	0	0	1	0	0	0	1

thus $(P_1 \wedge P_2) \Rightarrow c$ is a tautology and hence $P_1 \wedge P_2 \vdash c$ is a valid argument.

Q Determine whether the following argument is valid or not

$$\begin{array}{l} p \Rightarrow \neg q \\ r \Rightarrow q \\ r \\ \hline \therefore \neg p \end{array}$$

p	q	r	$\neg q$	P_1: $p\Rightarrow \neg q$	P_2: $r \Rightarrow q$	P_3: $P_1 \wedge P_2 \wedge P_3$	$\neg p$	$P_1 \wedge P_2 \wedge P_3 \Rightarrow \neg p$
0	0	0	1	1	1	0	1	1
0	0	1	1	1	0	0	1	1
0	1	0	0	1	1	0	1	1
0	1	1	0	1	1	1	1	1
1	0	0	1	1	1	0	0	1
1	0	1	1	1	0	0	0	1
1	1	0	0	0	1	0	0	1
1	1	1	0	0	1	0	0	1

Hence $P_1 \wedge P_2 \wedge P_3 \Rightarrow C$ is a tautology and hence $P_1 \wedge P_2 \wedge P_3 \vdash C$ is a valid argument

Rule of Inference for propositional Calculas:-

Rule of Inference	Tautology Form	Name
① $\dfrac{p}{\therefore\ p \vee q}$	$p \Rightarrow (p \vee q)$	Addition
② $\dfrac{p \wedge q}{\therefore\ p}$	$(p \wedge q) \Rightarrow p$	Simplification
③ $\dfrac{p \quad q}{\therefore\ p \wedge q}$	$((p) \wedge (q)) \Rightarrow (p \wedge q)$	Conjunction
④ $\dfrac{p \Rightarrow q \quad p}{\therefore\ q}$	$[(p \Rightarrow q) \wedge p] \Rightarrow q$	Modus Ponens
⑤ $\dfrac{p \Rightarrow q \quad \neg q}{\neg p}$	$[(p \Rightarrow q) \wedge (\neg q)] \Rightarrow \neg p$	Modus Tollens
⑥ $\dfrac{p \Rightarrow q \quad q \Rightarrow r}{\therefore\ p \Rightarrow r}$	$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$	Hypothetical Syllogism
⑦ $\dfrac{p \vee q \quad \neg p}{\therefore\ q}$	$[(p \vee q) \wedge \neg p] \Rightarrow q$	Disjunction Syllogism
⑧ $\dfrac{(p \Rightarrow q) \wedge (r \Rightarrow s) \quad p \vee r}{\therefore\ q \vee s}$	$[(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r)] \Rightarrow q \vee s$	Constructive Dilemma

$$\begin{array}{l} (p \Rightarrow q) \wedge (r \Rightarrow s) \\ \neg q \vee \neg s \\ \hline \therefore \neg p \vee \neg r \end{array}$$

$[(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (\neg q \vee \neg s)] \Rightarrow (\neg p \vee \neg r)$ Destructive Dilemma.

<u>Exp.</u> Apply the rules of inference, show that the following is valid.

$$\begin{array}{c} (P \vee Q) \Rightarrow S \\ S \Rightarrow R \\ \neg(R \vee Q) \\ \hline \therefore \neg P \end{array}$$

<u>Sol^n.</u>

① $(P \vee Q) \Rightarrow S$

$S \Rightarrow R$

$\underline{\neg R} \wedge \underline{\neg Q}$ (apply Demorgans Law)

② $(P \vee Q) \Rightarrow S$

$S \Rightarrow R$

$\neg R$ (apply Simplification Rule 2)

③ $(P \vee Q) \Rightarrow S$ ~~Contrapositive $S \Rightarrow R \Leftrightarrow \neg R \Rightarrow \neg S$~~

$S \Rightarrow R$ (apply ~~Modus Tollens~~)

$\neg R$

④ $(P \vee Q) \Rightarrow S$

$\neg S$ (apply Modus Tollens)

⑤ $\neg(P \vee Q)$ apply Modus Tollen

⑥ $\neg P \wedge \neg Q$

⑦ $\therefore \neg P$ Rule ② Simplification.

Q. Check the validity of the argument

$$p \Rightarrow r$$
$$\neg p \Rightarrow q$$
$$q \Rightarrow s$$
$$\therefore \neg r \Rightarrow s$$

Sol^n.

① $\begin{cases} \neg r \Rightarrow \neg p \\ \neg p \Rightarrow q \end{cases}$ Contrapositive if $p \Rightarrow r$, $\neg r \Rightarrow \neg p$

$q \Rightarrow s$

② $\neg r \Rightarrow q$ Rule 6 in first two lines

$q \Rightarrow s$

③ $\therefore \neg r \Rightarrow s$ " "

Q. Test the validity of the following argument

$$p \Rightarrow q$$
$$r \Rightarrow \neg q$$
$$\therefore p \Rightarrow \neg r$$

Sol^n.

① $p \Rightarrow q$

$q \Rightarrow \neg r$ Contrapositive to second line
$r \Rightarrow \neg q \Leftrightarrow q \Rightarrow \neg r$

② $p \Rightarrow \neg r$ again Contrapositive

Q Check the validity of following argument

If the races are fixed or casinos are crooked, then the tourist trade will decline. If the tourist trade decreases, then police will be happy. The police force is never happy. Therefore the races are not fixed.

p: The races are fixed
q: The casinos are crooked
r: The tourist trade will ~~decline~~ decrease.
s: Police force is happy

The argument now becomes

(i) $(p \vee q) \Rightarrow r$
(ii) $r \Rightarrow s$
(iii) $\neg s$

$\therefore \neg p$

Soln. ① $(p \vee q) \Rightarrow r$

$\left.\begin{array}{l} r \Rightarrow s \\ \neg s \end{array}\right\}$ apply Modus Tollens

② $(p \vee q) \Rightarrow r$ ← apply contrapositive

$\neg r$

③ $\neg r \Rightarrow \neg (p \vee q)$ apply Modus Ponen

$\neg r$

④ $\neg (p \vee q)$

⑤ $\neg p \wedge \neg q$ Simplification Rule ②

⑥ $\therefore \neg p$

Q. Check the validity of the following arguments.

If there was a ball game, then travelling was difficult. If they arrived on time, then travelling was not difficult. They arrived on time. Therefore there was no ball game

Solⁿ.

Let p: There was a ball game
q: Travelling was difficult
r: They arrived on time

Now arguments in symbolic form

$$\begin{array}{l} p \Rightarrow q \\ r \Rightarrow \neg q \\ r \\ \hline \therefore \neg p \end{array}$$

① $p \Rightarrow q$
$r \Rightarrow \neg q$ } apply Modus Ponen
r

② $p \Rightarrow q$
$\neg q$

③ $\neg q \Rightarrow \neg p$ Contrapositive of step 1
$\neg q$

④ $(\neg p)$ apply Modus Ponen on above step.

Predicate Calculus

The propositional calculus does not allow us to represent many of the statements that we use in mathematics & in every day life. Predicate calculus is the Generalization of Propositional logic (calculus).

The part of a statement that follows the subject is called a 'predicate'.

ep.1 Consider the statement

"Every person who is 18 years or older, is eligible to vote"

This statement can't be adequately (पर्याप्त रूप से) satisfactory expressed using only propositional Logic.

It would be easier if the statement were refering to a specific person. But it is not the case & the statement applies to all people who are 18 years or older, we are stuck. so we can solve it using Predicate calculus.

Exp2. Consider the proposition P_1 and P_2

P_1: Ram is a MCA.

P_2: Shyam is a MCA.

As a proposition there is no relation between P_1 & P_2 but they have some common Part. We can replace the two proposition by a single statement "x is a MCA". Hence the common feature "is a MCA" is a predicate.

→ The Predicate P can be consider as a f^n, it tells the truth value of the statement P(x) at x."

Thus a part of declarative scentence describing the properties of an object or relation among objects is known as a predicate.

We can describe this as P(x) where P denotes the predicates and x is a variable or predicate variable. for exp. "x is greater than 3". It has two parts. The first part, is variable x (subject) & second is "is greater than 3".

Universe of Discourse:-

The universe of discource (or domain) of a predicate variable is the set of all possible values that may be substituted in place of variables.

For exp.

P(x): x is a student, can be the set of all human names.

Quantifier: Quantifiers are words that refers to quantities such as "some" or "all" and indicate how frequently a certain statement is true. Quantifiers are classified into two types

1. Universal Quantifiers
2. Existential Quantifiers.

① Universal Quantifier ($\forall$)

Let P(x) be a statement defined on universe of discourse A. Then the universal quantification of a predicate P(x) is the statement "for all values $x \in A$, P(x) is true" & is denoted by in either of the following ways.

(a) $\forall x \in A : P(x)$ (b) $\forall x \, P(x)$

(c) $P(x), \forall x \in A$ (d) P(x) is true for all $x \in A$

↑

(universal quantifier)

The statement "x is greater than 3" can be denoted by $P(x)$ where P denotes the predicate "is greater than 3" and x is variable. The Predicate P can be considered as f^n. It tells the truth value of $P(x)$. Once a value has been assigned to the variable x, the statement $P(x)$ ~~is~~ becomes proposition with true or false value.

eg. The proposition $\forall x \in N;\ x+3 > 3$ is true since

$$\{x : x+3 > 3\} = \{1, 2, 3, 4, ---\} = N.$$

Existential Quantifier :- ($\exists$)

Let $P(x)$ be a propositional functional defined on a set A. Then the existential quantification of a predicate $P(x)$ is the statement "there exists at least one value in A such that $P(x)$ is true" & is denoted by the following ways.

(1) $\exists x \in A : P(x)$ (2) $\exists x ; P(x)$

(3) There exist is an x such that $P(x)$ (4) $P(x)$ is true for some $x \in A$.

The symbo $\exists$ which reads 'there exist', or 'for some' or 'for at least one' is called the existential quantifiers. Can also be represented as

$$[x : x \in A,\ P(x)] \neq \phi$$

eg. The proposition $\exists x \in Z ; -1 \leq x \leq 1$ is true since

$$\{x : -1 \leq x \leq 1\} = \{-1, 0, 1\} \neq \phi$$

eg " " "$\exists x \in N$: "$x+8 < 2$" is false. since

$$\{x \in N ;\ x+8 < 2\} = \phi .$$

Q. Express the statements as logical expressions.

① Every student spends more than 5 hours every weekday in class

② There is a student who spends more than 5 hours every weekday in class.

Sol^n. Let $P(x)$ (variable) denote the statement "x spends more than 5 hours every weekday in class" (Predicate). Let universe of discourse for x is the set of students. Then

(i) $\forall x\ P(x)$ (2) $\exists x\ P(x)$

& P is f^n function that tells the truth value of $P(x)$
→ statement
x is variable or subject

mortal (नश्वर)

Q. Let $M(x)$: x is man
$N(x)$: x is mortal
$A(x)$: x is integer
$B(x)$: Either positive or negative

Express the following using quantifiers.
(a) All men are mortal
(b) Any integer is either positive or negative

Sol^n: The Given statement can be written as
For All x, if x is a man, then x is mortal

$$(\forall x)\ (M(x) \Rightarrow N(x))$$ or

(b) For all x, if x is an integer then x is either positive or negative.

$$(\forall x)\ (A(x) \Rightarrow B(x))$$

Q.* Let $A(x)$: x is a student
$B(x)$: x is clever
$C(x)$: x is successful.

Express the following using quantifiers
(a) There exists a student
(b) Some students are clever
(c) Some students are not ~~ef~~ successful

Universe of discourse is set of students.

(a) $(\exists x)\ (A(x))$

(b) There exist an x such that x is a student and x is clever.

$$(\exists x)\ (A(x) \wedge B(x))$$

OR There $\exists x \in A,\ B(x)$

(C) There exist an x such that x is student and x is not successful.

$$(\exists x)\ (A(x) \wedge \neg C(x))$$

Q. Express the statement, "All clear expressions are satisfactory" as logical expressions.

Ans.

$\rightarrow$ Proposition

Let $P(x)$: x is clear explanations.

$Q(x)$: x is satisfactory.

then, the given statement can be expressed as.

For all x, if x is clear expression than x is satisfactory

$$(\forall x)\,(P(x) \Rightarrow Q(x))$$

Notation. When all the elements in the universe of discourse can be listed - say $x_1, x_2, x_3, --- x_n$ then.

① The universal quantification $\forall x\, P(x)$ can be written as

$$P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge ----- \wedge P(x_n) \quad — ①$$

because ① is true if and only if each $P(x_i)$ for $i=1,2,3,--n$ is true.

② The existential quantification $\exists x\, P(x)$ is same as the disjunction

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee ---- \vee (P(x_n) \quad — ②$$

as ② is true if and only if at least one of $P(x_i)$ is true.

Q. Let $A = \{a, e, i, o, u\}$ and $P(x)$: x is a vowel. Then express the ~~logical~~ ~~ex~~ statement in logical expression.

"a is a vowel and e is a vowel and i is a vowel and o is a vowel and u is a vowel".

Sol^n Let ~~pre $x_1 = a$, $x_2 = e$, $x_3 = i$, $x_4 = o$, $x_5 = u$~~

Now $P(x)$: x is a vowel.

$P(a)$ = a is a vowel	then. $P(a) \wedge P(e) \wedge P(i) \wedge P(o) \wedge P(u)$
$P(e)$ = e is a vowel	or $(\forall x)(P(x))$
$P(i)$ = i " "	or $\forall x \in A : P(x)$
$P(o)$ = o " " "	
$P(u)$ = " " "	

Q. Let $P(x,y) \equiv x = y+5$. What are the truth values of propositions $P(2,3)$ and $P(6,1)$?

Solⁿ. $P(2,3) = 2 = 3+5$

$2 = 8$ which is false

Hence truth value of $P(2,3) =$ is false

$\therefore$ $P(6,1)$ $6 = 1+5$

$6 = 6$

Truth value of $P(6,1)$ is true.

Quantifiers

Statements	When True	When False
$\forall x, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Q. Let $P(x)$ denotes the statement "$x+2>5$" and the universe of discourse A consists of the positive integers not exceeding 4. What is the truth value of

(i) $\forall x \in A: P(x)$

(ii) $\exists x \in A: P(x)$

Solⁿ. Here $A = \{1,2,3,4\}$

$P(1): 1+2>5$ $\quad P(2): 4>5$

$3>5$ $\quad P(3): 3+2=5>5$

Proposition $P(4): 4+2=6>5$ are false because in universal quantification each proposition must be true.

(ii) $\exists x \in A: P(x)$

$P(1): 3>5$ $\quad P(2): 4>5$ $\quad P(3): 5>5$ $\quad P(4): 6>5$ true

truth value is true.

Negation of Quantified Statement

Consider the statement

"All the students in this class are intelligent"

Its negation reads: —— (i)

"It is not the case that all the students in the class are intelligent" —— (ii)

or There ~~exist~~ is a student in the class who is not intelligent. —— (iii)

Let $P(x)$: x is intelligent and S denotes the set of all students in a class, then statement (i) can be expressed as

$\forall x \in S : P(x)$

statement (ii) & (iii) can be expressed as

$\neg(\forall x \in S : P(x))$ &

$\exists x \in S : \neg P(x)$

but (ii) and (iii) are equivalent.

$\therefore \neg(\forall x \in S : P(x)) = \exists x \in S : \neg P(x)$

$(\forall x)\ S(x) \Rightarrow P(x)$

$\neg[(\forall x)\ S(x) \Rightarrow P(x)]$

$\exists x\ \neg(\neg S(x) \vee P(x))$

$\exists x\ \underline{S(x)} \wedge \underline{\neg P(x)}$

There exist an x such that x is a student in the class and x is not intelligent

OR

There is a student in the class who is not intelligent.

Equivalent forms of Universal and existential statement

Let the universe of discourse is set of integers.

"$\forall$ x, if $\overbrace{x \text{ is an integer}}^{P(x)}$ than $\overbrace{x \text{ is rational}}^{Q(x)}$"

"$\forall$ integers x, x is rational"

Both have same translation "All integers are rationals".

Infact, a statement

$\forall x \in U$, if $P(x)$ then $Q(x)$

$\forall x \in U$, $P(x) \Rightarrow Q(x)$ —— ① First form

can be rewritten as

$\forall x \in D, Q(x)$

by narrowing U to be domain D, where D is the truth set of P(x) (containing of all values of variable x that make P(x) true).

OR D is the set of all x for which $P(x)$ is true.

Conversely, a statement of the form

OR $\forall x \in P, Q(x)$ —— ② second form.

can be written as

$\forall x$, if x is in P, then $Q(x)$.

for:

Exp. "All square are rectangle" Convert it into both the form

let $P(x)$: x is a square

$Q(x)$: x is a rectangle

statement

For all x, if x is a square then x is a rectangle

$(\forall x)(P(x) \Rightarrow Q(x))$

OR

$\forall x \in P, Q(x)$

(Next Exp. is on previous page)

Equivalent forms of Existential Quantifier

Similarly, a statement of the form

"$\exists x$ such that $P(x)$ and $Q(x)$"

Can be written as

"$\exists x \in D$ such that $Q(x)$"

where D is the set of all x for which $P(x)$ is true.

universe of discourse.

Q Write the negation of the statement:

(i) $\forall x \in R,\ x > 3 \Rightarrow x^2 > 9$

(ii) $\forall$ set A, if $A \subseteq R$ then $A \subseteq Z$

State which one is true — the statement or its negation.

Ans. (i) let $P(x) = x > 3$ & $Q(x) = x^2 > 9$

statement

then the given statement is ~~$\forall x \in R$~~

$$\forall x : P(x) \Rightarrow Q(x) \quad \text{——} ①$$

negation of eq^n ① is

$$\neg(\forall x : P(x) \Rightarrow Q(x))$$

$$\equiv \exists x : \neg(\neg P(x) \vee Q(x))$$

$$\equiv \exists x : P(x) \wedge \neg Q(x)$$

There exists a real number x such that $x > 3$ and $x^2 \leq 9$".

So statement ① is true and its negate is not true

(ii) Let $R(A) = A \subseteq R$ and $S(A) = A \subseteq Z$

then

$$\forall A : R(A) \Rightarrow S(A) \quad \text{—— (2)}$$

Negation of (2)

$$\neg(\forall A : R(A) \Rightarrow S(A))$$
$$\equiv \exists A : \neg(R(A) \Rightarrow S(A))$$
$$\equiv \exists A : \neg(\neg R(A) \vee S(A))$$
$$\equiv \exists A : R(A) \wedge \neg S(A)$$

There exists a set A such that $A \subseteq R$ and $A \nsubseteq Z$.

So Statement (2) is false and its negate is true.

Q. Express the following statement as logical expression
(i) All birds can fly
(ii) Some birds can fly
(iii) Not all birds can fly.

Solⁿ. Let us denote

$B(x) = x$ is a bird

$F(x) = x$ can fly.

and universe of discourse U is the set of all birds. then

(i) $\forall x : B(x) \Rightarrow F(x)$

or $\forall x \in B, F(x)$

(ii) There exists a x, such that x is a bird and x can fly.

$$\exists x, [B(x) \wedge F(x)]$$

or

$$\exists x \in B, F(x)$$

(iii) $\neg(\forall(x) : B(x) \Rightarrow F(x))$

or $\exists x : [B(x) \wedge \neg F(x)]$

or $\exists x \in B, \neg F(x)$

Q. Write the negation of the following statement.

(i) Every complete ~~bipet~~ bipartite graph is not planer

(ii) For each integer x, if x is even, then x^2+x is even.

(iii) Some number are not real.

~~Ans~~ (iv) All students live in the dormitories.

Soln. (i) Every complete bipartite graph is not planner.

let universe of discourse U, consist of all graphs.

$C(x)$: x is complete

$B(x)$: x is bipartite

$P(x)$: x is planner

then, the given statement can be expressed as

$$\forall x : [C(x) \wedge B(x) \Rightarrow \neg P(x)]$$

negation is

$$\neg(\forall x : ((C(x) \wedge B(x)) \Rightarrow \neg P(x))$$

$$\exists x : \neg(C(x) \wedge B(x)) \Rightarrow \neg P(x))$$

$$\exists x : \neg(\neg((C(x) \wedge B(x)) \vee \neg P(x))$$

$$\exists x : [C(x) \wedge B(x) \wedge P(x)]$$

* Some complete bipartite graphs are planner

OR There exists an x such that x is comple, ~~and~~ bipartite ~~graph~~ and planner graph.

(ii) Let the universe of discourse be the set of integers, Z and

$A(x)$: x is even
$B(x)$: x^2+x is even

then, the given statement is

$$\forall x \in Z, [A(x) \Rightarrow B(x)]$$

& its negation

$$\neg(\forall(x): [A(x) \Rightarrow B(x)])$$
$$\exists x : [A(x) \wedge \neg B(x)]$$

"There exists ~~at~~ a x in Z such that x is even and x^2+x is not even."

(iii) Let the universe of discourse be the set of real numbers

Let $N(x)$: x is a number
$R(x)$: x is real.

then, Some numbers are not real canbe expressed as

$$\exists x : (N(x) \wedge \neg M(x))$$

& its negation is

$$\forall(x) : \neg(N(x) \wedge \neg M(x))$$
$$\cancel{\forall(x) : N(x)}$$
$$\forall(x) : \neg\{N(x) \vee M(x))$$
$$\forall(x) : (N(x) \Rightarrow M(x))$$

"All numbers are real numbers"

(iv) All students lives in the dormatories.

let the universe of discours is the students.

Let $S(x)$: x is student

$D(x)$: x lives in dormitories.

then $\forall(x) : [S(x) \Rightarrow D(x)]$

and its negation

$$\exists x : \neg(S(x) \Rightarrow D(x))$$

$$\exists x : [S(x) \wedge \neg(D(x))]$$

There exists a x, such that x is a student and x does not live in Dormitories.

OR Some students do not live in Dormitories.

OR At least one student who do not ~~leave~~ live in the dormitories

By - KKV Sir